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INTENSITIES OF K-SERIES X-RAYS FROM THIN TARGETS

By D. L. Webster, H. Clark, R. M. Yeatman and W. W. Hansen

Department of Physics, Stanford University

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I. *Introduction.*—This paper is a preliminary report on a study of the probability that a cathode ray of known energy, invading an atom of silver, will remove a K electron from it.

On the experimental side, a comparison of such probabilities, for any two cases, is made most readily by comparison of the intensities of the x-rays of the Kα line. The x-rays to be compared, however, must be obtained from a target in which all the cathode rays striking any atoms of silver have the same energy. The ordinary, thick metal targets do not satisfy this condition, because most of the atoms are struck only by cathode rays which have previously lost much of their energy in other atoms. The target used for this purpose must therefore be so thin as to allow almost all of the cathode rays to go through it with no appreciable loss of energy.

Information about x-rays from such extremely thin targets has been collected from various sources, some of which are listed in footnotes 1 to 9, inclusive. Most of this information, however, deals with the continuous spectrum only; and such of it as applies to K-electron ionization depends on theoretical deductions from experiments on thick targets. In view of the importance of the problem of the mechanism of K-electron ionization, direct evidence from thin targets is highly desirable, and we have therefore obtained such evidence in experiments to be reported in this paper.

II. *Apparatus.*—The central feature of the apparatus used here is naturally the thin target, which is made on the principle first suggested by Davis,[3] of depositing a thin layer of silver on a block of another element. The first block used was copper, but investigation[10] showed that the indirect rays from the silver, due to fluorescence under the action of rays from the copper, might be strong enough to cause appreciable errors, and the copper block was therefore replaced by beryllium. As this block

was set, the cathode rays struck the silver at 90°, and the x-rays used left it at 3°.

The manner of deposition of the silver is important, because the film must be extremely thin, not only on the average, but everywhere. Crystal growth must therefore be prevented as far as possible. The first films, deposited electrolytically on copper from $KAg(CN)_2$, were altogether too crystalline, until they were polished; and even after that, we could not be sure they had not been thickened in spots, in minute depressions in the copper block. Films deposited by evaporation from a silver wire, heated in a high vacuum, were likewise defective, but more rapid deposition, from a drop of molten silver in a molybdenum trough, seemed to remedy this defect. Not only were these rapidly deposited films extremely good mirrors, but they became opaque (as seen when deposited on copper or celluloid) at thicknesses of about 50 Å. Such opacity would not be expected in layers of isolated crystals with open spaces between them, and so we may infer that the silver begins to deposit everywhere before the average thickness reaches 50 Å, and therefore that the irregularities in thickness of any of these films are probably smaller than the 1000 Å or so required to retard cathode rays appreciably.

The best thickness to use is a matter of compromise. If the film is too thick, some of the cathode rays will be retarded before producing K-electron ionization; if it is too thin, the intensity of the silver lines will be so small compared with that of the continuous spectrum of beryllium, that accurate measurements cannot be made. To be on the safe side, we made the first film very thin, and others with progressively increasing thickness.

During experiments with the bare beryllium, and again with each film, the focal spot became strongly colored under the action of the cathode rays, but the silver lines maintained constant intensities, showing that no silver was lost. The coloration of each film might tempt one to polish the surface before depositing the next film, but the danger of thus producing thick spots (thick as compared to 100 Å) made it seem better to make each new film, by addition to the last, without touching the surface at all. In each case, the colored focal spot was hidden by the new silver quickly enough to show that the silver was depositing at about the same rate as it would on a surface of polished copper.

Estimates of the thicknesses of these films are useful for several purposes, but need not be exact. The best method we have found for obtaining them is based on a theoretical equation[1,2] confirmed qualitatively for the continuous spectrum by Duane[8] and Nicholas[9] and for the line spectrum by the present experiments. This equation connects the line intensity $i\Delta x$, from a thin target of thickness Δx, with the intensity I, from a thick target. This I must be the intensity of direct rays only, not counting

the indirect rays mentioned above. Neglecting absorption of x-rays in the target, the equation is

$$i = \frac{b}{2V} \cdot \frac{dI}{dV} \tag{1}$$

where V is the tube voltage and b is the constant of the Thomson-Whiddington law,

$$V^2 - V_s^2 = bs. \tag{2}$$

Empirically, it has usually been found possible to represent I approximately by

$$I = \text{const.} \times (V - V_k)^n \tag{3}$$

where V_k is the K excitation voltage and n is a constant, which may be as low as 1.5 with large target absorption or as high as 2 with small absorption.[7,11] If the ratio $i\Delta x/I$ is found by experiment, and called β, then

$$\Delta x = \frac{2V\beta}{bn}(V - V_k). \tag{4}$$

To find β, the line intensity from each target may be compared with the continuous background (by the ratio of line area to continuous-spectrum ordinate); here the continuous background for the thin target is found to be hardly any stronger than that from the bare beryllium, and may therefore be assumed to be only $^4/_{47}$ as strong as with the thick silver target.

The film thicknesses, estimated by the above method, are as follows:

Film	*a*	*b*	*c*	*d*	*e*
Thickness	30	50	80	170	280 Å

While each film was being deposited, the temperature of the molten silver was kept reasonably constant by heating the molybdenum trough at constant voltage. It is of interest to note that the thicknesses of the films as estimated above are roughly proportional (within 5 or 10%) to the total deposition times. It seems therefore that under the conditions used, the silver is deposited at an approximately uniform rate.

Another check on these data was obtained by plotting a spectrum of rays from a standard Coolidge tube, passing through some of the beryllium and then through the silver film. In such a spectrum, the effect of the silver absorption edge is to produce a discontinuity independent of the absorption in the tube walls or the beryllium. The probable error of this method is at least 100 Å, but the figure obtained for film *e* was 270 Å, the agreement with the 280 Å found by the other method being probably a matter of coincidence.

A fourth source of information about film thicknesses is given by the fact that if appreciable retardation of the cathode rays occurred, it would make the relative change in i for a given change in V greater for thick films than for thin, whereas experiment shows no such difference. This evidence is especially valuable as showing that even the thickest of these films gives the same form for the function i as an infinitely thin film.

The rest of the apparatus was all as described in previous papers,[2] except that the rays coming through the spectrometer were limited by

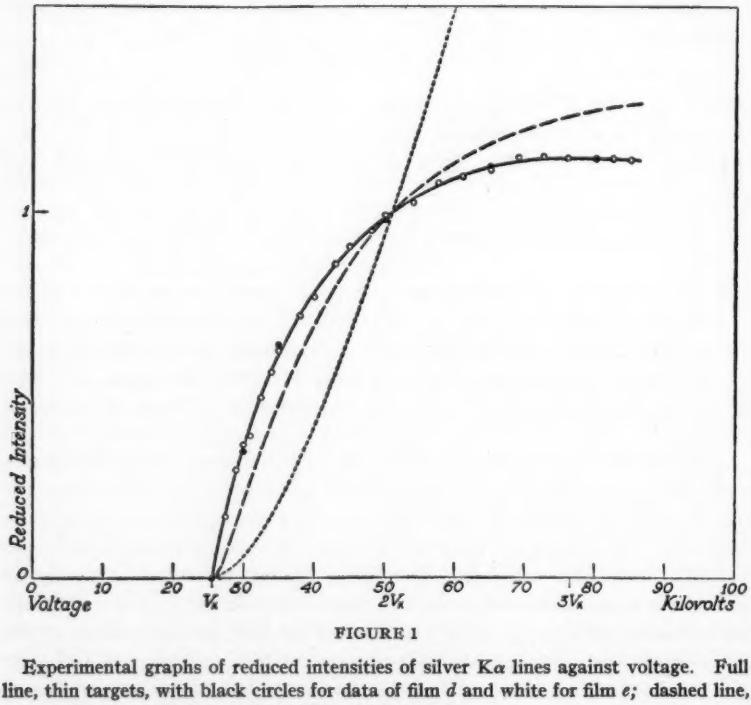

FIGURE 1

Experimental graphs of reduced intensities of silver Kα lines against voltage. Full line, thin targets, with black circles for data of film d and white for film e; dashed line, target of medium thickness; dotted line, thick target.

one slit only, and that that slit was wide enough to include all α-line rays striking the crystal at the reflecting angle. These are the conditions required for any accurate comparisons of line intensities at different voltages. The resolving power was sufficient to insure the complete absence of any line radiation over a region of reasonable width between the α and β lines. This permits accurate graphical interpolation for the continuous-spectrum intensities under the lines. At the same time, the

resolving power was low enough to give α peaks, with nearly flat tops, parallel to the continuous-spectrum graphs. This insures accuracy in the comparison of α-line intensities, by the subtraction of the ordinates of the interpolated continuous-spectrum graphs from the ordinates in the middles of the flat tops of the peaks.

The magnification of relative errors, given by such subtractions, is serious if the films are so thin as to make the peaks stand but little above the continuous-spectrum graphs. Films *a*, *b* and *c* suffered seriously from

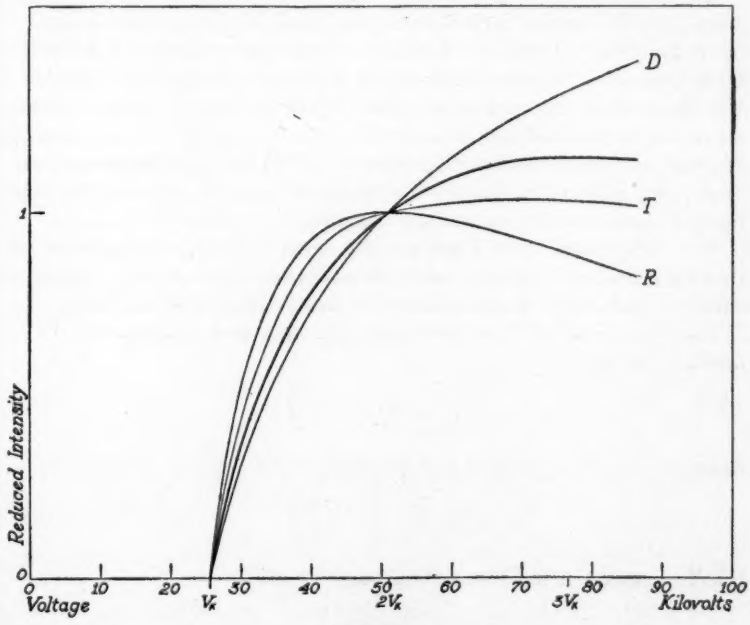

FIGURE 2

Comparison of experimental and theoretical graphs of thin-target intensities.

this defect, their line radiations being less than 20% of total radiations at that wave-length. In film *d*, however, this percentage was usually 20 to 30%, and in film *e*, 30 to 40%.

III. *Experimental Data.*—To compare results from these films, they are plotted in figure 1, in terms of what we may call the "reduced intensity," *j*, the ratio of *i* at any voltage *V* to its value at $2V_K$. Because of the difference in accuracy, just noted, points are indicated for films *d* and *e* only, and the graph is drawn for them.

To illustrate the contrast between this graph and corresponding ones

for thick targets, we have included also two other graphs. One is for a target about 10,000 Å thick and probably non-uniform, plated on copper and polished; the other is for a similar target at least 25 times as thick, which stops both cathode rays and K-series x-rays, and is therefore a "thick target," in the sense in which we have used that term here. The relation between this last graph, for which the n of equation (2) is about 1.5, and the thin-target graph is like their theoretical relation, (1), qualitatively but not quantitatively.

It must be noted that the ordinates of the graphs in figure 1 refer to total silver Kα-line radiations, including both the rays produced directly by cathode-ray ejection of K electrons and those produced indirectly by fluorescence. For the thick target, previous investigation[10] has shown that the indirect rays contribute about 35% of the intensity at all voltages; for thin targets, approximate calculations give a rising percentage: 0.002% at 35 kv., 0.03% at 50, 0.10% at 65 and 0.21% at 80. For present purposes, and within the limits of error of the present experiments, these indirect rays from thin targets are negligible.

IV. *Comparison with Theories.*—In figure 2 the heaviest line is the thin-target intensity graph, taken directly from figure 1 for comparison with the predictions of such theories as we have found in the literature.

The graph marked D is for Davis'[13] theory, which predicts for the reduced intensity,

$$j_D = 2\left(1 - \frac{1}{U}\right) \tag{5}$$

where $U = V/V_K$. Graph R is for Rosseland's[13] theory, which gives

$$j_R = \frac{4}{U}\left(1 - \frac{1}{U}\right). \tag{6}$$

Finally, graph T is for Thomas'[14] theory, for which

$$j_T = \frac{3 + k}{1 + k + U} \cdot \frac{1 + \frac{2}{3}k - \frac{1}{U} - \frac{2}{3}\frac{k}{U^2}}{\frac{1}{2}(1 + k)} \tag{7}$$

where k is the ratio of the kinetic energy of the K electron, in a Bohr orbit, to the ionization energy. The exact value to use for this is not certain, but the values of j_T are surprisingly insensitive to changes in k, even so far as down to $k = 0$, so we have used the value 1.278, given by the kinetic energy of a simple hydrogenic atom with a nuclear charge of 47.

For thick targets, the main features of the line-spectrum result so much from the laws of cathode-ray penetration, target absorption, etc., that all three of these theories give predictions nearly alike, and it is difficult to

decide among them by experiment. Here, with thin targets, the differences come to light more readily. All these theories are qualitatively right, in that the curvatures of their graphs are downward, in striking contrast to the upward curvatures of thick target graphs. In quantitative detail, however, while it may be too early in this work for definite estimates of experimental errors,[15] it is our opinion at present that none of the theories are strictly accurate. With regard to the best means of progress on this subject it seems reasonable to expect that here, as in so many other branches of atomic physics, we must look to the wave mechanics.

Jönsson,[7] from his experiments on the Kα lines of aluminum and the Lα_1 of silver in thick targets, deduced by equation (1), with an absorption correction, some graphs for thin-target intensities. These graphs showed maxima at 2.6 V_K and 2.4 V_{L_1}, respectively, and then a steady decline.

For an empirical equation for j, three requirements must be satisfied. First, at $V = V_K$, $j = 0$. Second, at $V = 2V_K$, $j = 1$. Third, at high voltages the inverse-square law, as applied by Rosseland and Thomas, demands that j shall eventually become more and more nearly proportional to $1/V$; this requirement is also not far from agreeing with Jönnson's conclusions. These requirements are all met by a convenient equation with only one arbitrary constant, as follows:

$$j = 2(m + 2)\frac{1 - \frac{1}{U}}{m + U}. \tag{8}$$

If $m = 3$, this equation fits the present data with errors not exceeding 0.03 of a unit of reduced intensity, so that

$$j = 10\,\frac{1 - \frac{1}{U}}{3 + U}. \tag{9}$$

V. *Summary.*—1. Experimental data have been obtained for targets of silver, so thin as to cause no appreciable retardation of most of the cathode rays. These data give the relative intensity of K-series x-rays, and, therefore, the relative probability of direct K-electron ionization, as a function of cathode-ray energy, from the excitation energy to 3.3 times that amount.

2. The results show that the three theories now available all give predictions of the right sort for this range of energies, but none of them is exact.

3. An empirical equation (8) has been found for these intensities.

In conclusion, it may be noted that, while further work is much needed on the theoretical side of this problem, these data likewise make no claim

to finality, and it is our present intention to refine and extend this work as much as possible on the experimental side.

[1] B. Davis, *Physic. Rev.*, **9**, 64 (1917).
[2] D. L. Webster, *Ibid.*, **9**, 220 (1917).
[3] B. Davis, *Ibid.*, **11**, 433 (1918).
[4] D. L. Webster and A. E. Hennings, *Ibid.*, **21**, 312 (1923).
[5] H. A. Kramers, *Phil. Mag.*, **46**, 869 (1923).
[6] W. W. Nicholas, *Physic. Rev.*, **29**, 619 (1927).
[7] A. Jönsson, *Zeitschr. Physik*, **43**, 845 (1927).
[8] W. Duane, these PROCEEDINGS, **13**, 662 (1927); and **14**, 450 (1928).
[9] W. W. Nicholas, Amer. Physical Soc. meeting of April, 1928.
[10] D. L. Webster, these PROCEEDINGS, **13**, 445 (1927); and **14**, 330 (1928).
[11] D. L. Webster and H. Clark, *Ibid.*, **3**, 181 (1917).
[12] Note 10, also D. L. Webster and R. M. Yeatman, *J. O. S. A. & R. S. I.*, **17**, 248 and 254 (1928).
[13] S. Rosseland, *Phil. Mag.*, **45**, 65 (1923).
[14] L. H. Thomas, *Proc. Camb. Phil. Soc.*, **23**, 829 (1927).
[15] To avoid any false impression of accuracy, due to the self-consistency of the data in figure 1, two possible sources of systematic error should be kept in mind. One is recrystallization under cathode-ray impact, suggested by the coloring of the focal spot. The other is the production of x-rays by the cathode rays deflected so much in the beryllium as to strike the silver a second time. We doubt if the former of these errors is appreciable, but for the latter, even with such a light backing material as beryllium, there may be a correction of a few per cent.

THE SPECTRUM OF THE HYDROGEN MOLECULAR ION

BY C. J. BRASEFIELD*

RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

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Through the efforts of Richardson and others, considerable progress has been made in the analysis of the secondary spectrum of hydrogen. The analysis is however far from complete and is the more complicated by the fact that not only the diatomic hydrogen molecule but also the triatomic molecule, as well as their respective ions, are possible emitters of this spectrum. It is therefore highly desirable that some experiment be devised by which one emitter may be obtained to the exclusion of the rest. By means of a magnetic analysis of a luminous canal-ray beam in hydrogen, it has been shown[1] that at very low pressures the particles in the canal-ray beam are practically entirely hydrogen molecular ions. If these ions give rise to any radiation, not only would the analysis of the complete secondary spectrum be somewhat simplified, but also we should then have evidence of the band spectrum of the simplest conceivable molecule, namely, the hydrogen molecular ion.

The experimental arrangement used to produce the canal rays was quite similar to that used when the magnetic analysis was made. The iron pole pieces and magnetic shielding were found to be abundant sources of carbon monoxide and were thus removed. This left simply a conical sheet nickel anode and an oxide-coated strip filament, immediately behind which was the cathode, a disk of sheet nickel with a narrow slit in the center through which the canal rays passed. The filament and the cathode were at the same potential, while the potential of the anode was approximately 1000 volts. The current between filament and anode was kept constant at 20 milliamperes. At the lower pressures (below 0.005 mm.) this must have been largely electronic current, for it was unnecessary to put any resistance in series with the discharge to keep it stable. As the positive-ion current increased (with increased pressure), the discharge became more and more unstable, and above 0.005 mm. it was necessary to put 10,000 ohms in series to keep it stable.

The canal-ray beam was photographed about 0.5 cm. behind the cathode and perpendicularly to the direction of the beam. The spectrograph was a three-prism Steinheil instrument having an opening $f/3.0$ and giving a dispersion of 31.5 Å per mm. at H_β and 13.1 Å per mm. at H_γ. A 36-hour exposure on an Ilford Special Rapid Panchromatic plate showed no lines between H_α and H_β so the work was concentrated on the region between H_β and H_γ. For this region Cramer Hi-Speed plates were found to be by far the fastest available.

Unfortunately, the light emitted by the H_2^+ canal-ray beam comes not only from the radiation of the H_2^+ ion itself but also from radiation excited in the resting gas by the moving H_2^+ particles. However, it is quite easy to distinguish between the two effects. As the pressure increases, the number of H_2^+ ions formed in the ionization chamber and issuing from the cathode also increases. On the other hand, the mean free path of the H_2^+ ion is decreasing (as the pressure increases). Thus, as the pressure increases the number of H_2^+ ions which reach a certain point behind the cathode will increase to a maximum and then decrease again. With the point of observation at 0.5 cm. behind the cathode, it is easy to calculate that this maximum should come at about 0.007 mm. pressure. Hence, lines emitted by the H_2^+ molecule should increase to a maximum intensity at about 0.007 mm. pressure and then decrease in intensity as the pressure is raised further. On the other hand, lines arising from an excitation of the resting gas by the moving H_2^+ particles (as well as by their dissociation products, the H^+ ions) should increase constantly in intensity and more nearly as the square of the pressure.

A series of 30-hour exposures were taken of the canal-ray beam at 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.008 and 0.014 mm. pressure. Microphotometric records were made of the plates and then compared. It was

found that the lines appearing on the plates could be divided roughly into three groups. A large number increased rapidly in intensity with increasing pressure. Included in this group were practically all the lines of Richardson's and Allen and Sandeman's bands. The second group contained lines remaining approximately constant in intensity (above 0.005 mm. pressure), while the lines of the third group increased in intensity to a maximum between 0.005 and 0.008 mm. and then decreased again.

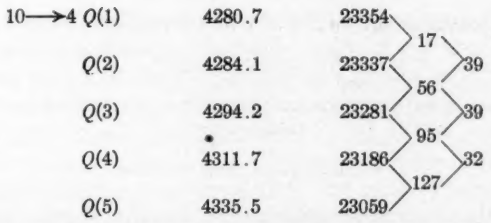
It is this third group in which we are interested, for these are the lines which are probably emitted by the H_2^+ ions. A table of their wave-lengths appears below:

5003.4	4790.4*	4596.7?	4479.7	4353.4*
4990.1	4786.2	4571.2?	4477.1	4335.5
4951.2?	4775.5	4558.8?	4430.3?	4312.9*
4942.0*	4730.8?	4547.2	4426.6	4311.7*
4937.8	4693.8	4541.1	4407.1	4308.6*
4912.3	4691.2	4529.7*	4404.6*	4294.2
4886.4	4678.1	4521.4	4396.9*	4284.1*
4802.4*	4674.5	4505.6	4394.3	4282.7*
4793.9	4620.8*	4489.2	4391.7	4280.7

Most of these lines are very weak compared to the lines in the first two groups, except when they are at maximum intensity. The wave-lengths are taken from Gale, Monk and Lee's table,[2] the discrepancy from the measured wave-lengths being in general not more than a few tenths of an Ångstrom unit. The lines followed by a question mark are so weak as to be doubtful; those followed by an asterisk show the predicted effect very markedly. λ 4620.8 is the only line of this group also claimed by Richardson's α, β,γ bands.

The writer has succeeded in arranging a number of these lines into three branches. Inasmuch as no other branches were found having the same heads, these are assumed to be Q branches.

MEMBER	λ	ν	1ST DIFF.	2ND DIFF.
8⟶4 $Q(1)$	4786.2	20888		
			19	
$Q(2)$	4790.4	20869		33
			52	
$Q(3)$	4802.4	20817		
9⟶4 $Q(1)$	4477.1	22330		
			13	
$Q(2)$	4479.7	22317		35
			48	
$Q(3)$	4489.2	22269		33
			81	
$Q(4)$	4505.6	22188		36
			117	
$Q(5)$	4529.7	22071		



It will be noticed that the bands degrade rather violently to the red which means that the initial moment of inertia is much larger than the final moment of inertia. This is what one would conclude if the H_2^+ molecule consisted of two protons bound by an electron whose orbit was perpendicular to the axis joining the protons. Moreover, such an unstable molecule would probably only be able to carry at most one quantum of vibrational energy. The bands given above are assumed to be the $0 \longrightarrow 0$ bands.

It is interesting to note that these series lie on the short wave-length side of the He^+ lines at 4859, 4542 and 4339. Because of the analogy which might be expected between H_2^+ and He^+, the corresponding electronic transitions have been assigned to these bands, namely, $8 \longrightarrow 4$, $9 \longrightarrow 4$ and $10 \longrightarrow 4$. The He^+ lines corresponding to the transitions $7 \longrightarrow 4$ and $11 \longrightarrow 4$ lie at 5411 and 4200, respectively, and are hence out of the region of observation. The only other He^+ line in the region of observation is at 4686 ($4 \longrightarrow 3$). A number of lines in the table are grouped in the neighborhood of this He^+ line but the writer was unable to order them into any series. Several other isolated branches besides the three given were found. However, no relation could be found between them so no especial importance was attached to them.

Finally, it is significant that the heads of these three bands are given by the simple Rydberg formula

$$\nu = A - \frac{4R}{(m + \mu)^2} \qquad m = 8, 9, 10$$

where $A = 27{,}743$, $R = 109678.3$ and $\mu = 0$ approximately. This value of A is approximately $4R/(3.98)^2$ which is in accordance with the assigned electronic transitions.

* NATIONAL RESEARCH FELLOW.

[1] C. J. Brasefield, *Phys. Rev.*, **31**, 215 (1928).

[2] H. G. Gale, G. S. Monk and K. O. Lee, *Astro. J.*, **67**, 89 (1928).

THE ABSORPTION SPECTRUM OF NITROGEN DIOXIDE

BY LOUIS HARRIS*

RESEARCH LABORATORY OF PHYSICAL CHEMISTRY, MASSACHUSETTS INSTITUTE OF TECHNOLOGY†

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Although the absorption spectrum of nitrogen dioxide has been the subject of some interest and study, few experiments have been reported with the gaseous phase under well-controlled concentration conditions of the single, NO_2, and the double, $(NO_2)_2$, molecules. A summary of recent experimental results obtained for the absorption of light by the system $2NO_2 = N_2O_4$, in the gaseous phase and in the ultra-violet and visible region of the spectrum, is herewith presented.

Warburg and Leithauser[1] found that nitrogen dioxide had several absorption maxima in the infra-red region. The maxima at $3{,}43\mu$ and $6{,}12\mu$ were ascribed to absorption by the NO_2; the maximum at $5{,}7\mu$ was ascribed to absorption by $(NO_2)_2$. Bell[2] found absorption in the visible due to NO_2 alone. Liveing and Dewar[3] ascribed the continuous absorption of the liquid to $(NO_2)_2$ and the banded structure of the liquid and of the vapor to NO_2. In a study of the emission spectrum of nitrogen dioxide, Zenneck and Strasser[4] found that the double, $(NO_2)_2$, molecules gave a continuous background in preference to lines and bands of the single, NO_2, molecules. Recently,[5] L. C. K. Carwile has measured the absorption bands of NO_2 from 3978 Å to 6323 Å with a twenty-one foot grating. The results of these investigators show with some certainty that the absorption in the visible region is due to single, NO_2, molecules alone. The following experiments agree with this conclusion and, furthermore, give the nature of the absorption spectrum of the single and double molecules in the ultra-violet region.

Nitrogen peroxide was prepared by heating $Pb(NO_3)_2$ in an all pyrex glass apparatus. The nitrogen dioxide was dried with P_2O_5; NO and other gases were pumped off; the nitrogen dioxide was then sublimed into quartz absorption vessels through a pyrex-quartz graded seal. The product was a pure white solid when cooled to $-30°C$.

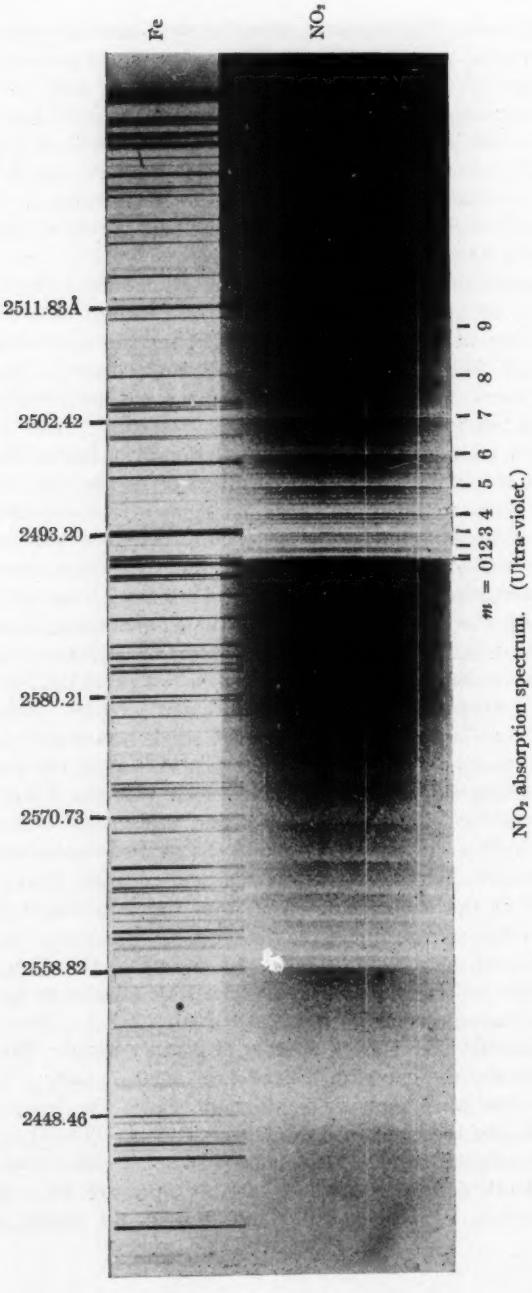
Two sets of absorption cells were used. Each cell was made entirely of clear crystal quartz and provided with plane polished windows. The absorption tubes of each cell were connected to a reservoir, containing the liquid nitrogen dioxide. The pressure of the gas was varied by changing the temperature of the reservoir. The ratio of single to double molecules was changed—at constant total pressure—by changing the temperature of the absorption tube. Cell No. 1 consisted of three absorption tubes of different lengths (200 mm., 20 mm. and 2 mm.), connected to

the same reservoir. A furnace, provided with quartz windows at each end, surrounded the three tubes. This design of cell proved very useful and time saving. Cell No. 2 consisted of a quartz tube 150 mm. long. The tube consisted of three sections, in tandem, each 50 mm. long, separated from each other by plane windows. The middle section was connected to a reservoir; the two end sections were evacuated. This cell was used essentially for measurements at low temperatures. The evacuated sections, projecting outside the cooling bath, obviated the condensation of water from the room on the windows.

A condensed under-water spark between aluminium electrodes served as a source of continuous ultra-violet light. A gas-filled bulb with a thick tungsten filament served as source of light for the visible region. The spectra were photographed with Hilger E_1 and Hilger E_2 spectrographs. The iron spark spectrum served as comparison spectrum. Exposures varied from two to twenty minutes.

The vapor pressure and equilibrium relationships showed that it would be difficult to obtain a large concentration of double, $(NO_2)_2$, molecules in the gas phase, without the accompaniment of a considerable amount of single, NO_2, molecules. The procedure adopted was therefore to investigate the absorption spectrum of the single molecules alone by maintaining the temperature of the tubes of cell No. 1 from 125° to 140°C., and varying the temperature of the liquid-containing-reservoir from −78°C. to +18°C. Incidentally, no absorption between 2200 Å and 6000 Å was observed with the reservoir at −78°C. and the 200 mm. tube, indicating the optical purity of the nitrogen dioxide. Next, a study was made of the absorption of mixtures of single and double molecules in the gaseous phase. For this series, the temperature of the reservoir was held constant while photographs were taken with the tubes at 20°C., 60°C. and 135°C. Finally, it was possible, using cell No. 2, to arrange conditions so that the absorption due to $(NO_2)_2$ alone was obtained. For these measurements, the reservoir temperature varied from −44°C. to −30°C., while the temperature of the tube was maintained two degrees higher than the temperature of the reservoir.

The results of the present investigation show that the absorption spectrum of single molecules, NO_2, consists of a large number of bands, many with fine structure extending from 6000 Å to 2250 Å. Extending from 2600 Å to 2250 Å, there are a number of distinct bands. Four of these bands, especially the one with a head near 2490 Å, show a well-defined structure. The accompanying photograph shows the band with head near 2490 Å and the band with head near 2460 Å. The lines are more diffuse in the 2460 Å band. These bands had not been reported and stand out so distinctly from one another that they may well serve as a key to the complicated, overlapping band spectrum in the visible region. A



NO_2 absorption spectrum. (Ultra-violet.)

number of weak, diffuse bands appear from 2700 Å to 3300 Å. From 3500 Å to 6000 Å a multitude of bands, many with fine structure, appear.

The absorption due to double molecules $(NO_2)_2$ is continuous. There seems to be two bands, one with a maximum at 3400 Å and another with a maximum further in the ultra-violet; the first absorption of this band occurs at 2400 Å. These bands make their first appearance for a pressure of 5 mm. Hg of $(NO_2)_2$ for a tube 50 mm. long. Increasing the pressure of $(NO_2)_2$ causes the bands to merge into one band which extends from about 4000 Å into the far ultra-violet.

The analysis of the absorption spectrum of NO_2 has been begun. A preliminary study seems to show that the bands belong to at least two electronic systems. The heads of some of the bands in the ultra-violet are separated by nearly the same interval (cm.$^{-1}$) as was found by Warburg and Leithauser[1] for the vibrational bands of NO_2 in the infra red region. The band with head near 2490 Å seems to be most suited for analysis of fine structure. This band consists of a number of strong fine lines whose disposition may be represented by the equation

$$Q_m = 40129{,}7 + 3{,}7\, m^2.$$

The units are in cm.$^{-1}$ and m varies from 0 to 12. This would indicate a "Q" branch, though no P or R branch corresponding to rotation about the same axis has been found until now. Between each of these strong lines, there are many fine lines, grouped as though coming to a head at each of the strong lines. The intermediate lines, measured, are from 0.06 Å to 0.20 Å apart and have not yet been ordered into their proper terms, though it would seem that they correspond to quantized rotation about a different axis than the strong lines referred to. The investigation is being continued.

Summary.—The absorption spectrum of single, NO_2, molecules consists of a large number of bands, many with fine structure, from 6000 Å to 2250 Å. A number of distinct bands, some with a well-defined structure, occur in the region from 2600 Å to 2250 Å. The absorption due to double molecules, $(NO_2)_2$, in the gaseous phase is continuous. Two bands, one with a maximum at 3500 Å and another with a maximum further in the ultra-violet, merge into one continuous band, at high pressures of $(NO_2)_2$, which extends from 4000 Å into the far ultra-violet.

This work was begun and certain phases of it completed at the Physical Chemistry Institute of the University of Zürich, Switzerland. I wish to thank Professor Victor Henri for extending the facilities of his laboratory and for his generous assistance and advice. I wish to thank, also, his assistant, Doctor M. C. Teves, for his assistance during the course of this work.

* NATIONAL RESEARCH FELLOW.
† Publication No. 215, Research Laboratory, Physical Chemistry, M. I. T.
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THE RELATION OF HYDROLYSIS TO THE VALIDITY OF BEER'S LAW[1,2]

BY R. C. GIBBS AND C. V. SHAPIRO[3]

DEPARTMENTS OF PHYSICS AND CHEMISTRY, CORNELL UNIVERSITY

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In the course of an extensive investigation on the absorption spectra of the phthaleins and related compounds,[5-10] a number of instances have been found in which Beer's law has failed to hold and in each case it has been possible to ascribe the deviations entirely to the action of hydrolysis or to an analogous process for alcoholic solutions. Inasmuch as the results are of some significance from both a chemical and physical point of view, it appears desirable to describe a few of the typical cases in some detail.

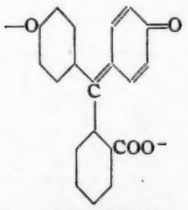
Beer's law may be stated thus: for a given thickness, the extinction, E, defined as $\log I_0/I$, is proportional to the concentration. It is based implicitly on the assumption that no change in the character of the absorbing centers occurs as the result of a variation in the concentration. Among the chemical factors that may affect this assumption, in that they are functions of the concentration, are association, dissociation, solvation and hydrolysis. It is also necessary to take into account on the physical side, as functions of the concentration, the mutual interaction of the electrostatic fields of the solute molecules, which gives rise to a form of Stark effect, and finally the change in refractive index of the solution. With the exception of hydrolysis, the above factors produce deviations from Beer's law which usually occur in the range of high concentrations and are, moreover, of limited magnitude, in that they become apparent only as relatively small changes in the intensity and position of existent bands.[4] On the other hand, hydrolysis occurs at low concentrations and, insofar as it favors the production of new types of absorbing centers, will result in a complete modification of the absorption spectrum.

Hydrolysis is a reaction which can be regarded as the reverse of neutralization and is found to occur in solutions of salts, either or both of whose components are derived from weak acids and weak bases. Phenolphthalein and its derivatives are all weak acids which are characterized by the

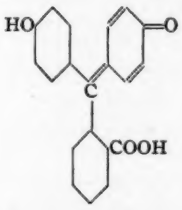
property of changing over to a tautomeric form when liberated from their dibasic salts. It is this property which renders them particularly suited for a spectroscopic study, since the absorption of the lactone form produced in the process of tautomerization is of a totally different type from that shown by the free ion. As salts of this class are largely dissociated in solution, the hydrolysis reaction can be expressed in the following reversible ionic equation:

$$(B^+ + A^-) + (H^+ + OH^-) = (B^+ + OH^-) + HA.$$

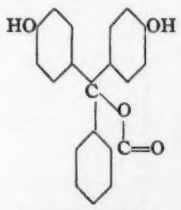
The ions of the weak acid combine with H^+ ions resulting from the partial dissociation of the water to form undissociated molecules of the free acid, which in the case of phenolphthalein rearranges to the lactone:



Dibasic Ion of Phenolphthalein



Free Acid —Unknown



Lactone Form of Phenolphthalein

It can readily be shown by an application of the law of mass action that the degree of hydrolysis is approximately inversely proportional to the square root of the concentration of the unhydrolyzed salt and is therefore a prominent factor at high dilutions. It is also evident from a consideration of the above reversible reaction that an increase of $(OH)^-$ ions, brought about by adding more alkali to the solution, will force back the hydrolysis with the production of more free ions of the dibasic salt.

Although the above discussion is based on aqueous solutions, it has been found experimentally that a similar course of reactions occurs in alcoholic solutions as well. This is illustrated in figure 1 for a series of alcoholic solutions of phenolphthalein.[5] Curve *A* represents the absorption of the neutral solution, characterized by a pair of bands at 3523 and 3613, while curve *C*, for a solution containing 232 molecules of potassium hydroxide to each molecule of phenolphthalein, shows the absorption of its colored ion with a strong band in the visual region at 1785 and three more bands in the ultra-violet which are evidently unrelated to the two bands of the neutral solution. Since phenolphthalein is a dibasic acid, the amount of alkali employed represents over one hundred times that necessary for salt formation. Under these conditions the type of absorption was found to be independent of the concentrations used and hence it may be assumed that hydrolysis had

been completely prevented and that curve C truly records the absorption characteristic of the dibasic ion of phenolphthalein. With a ratio of 23 molecules of potassium hydroxide to one of the phthalein, however, the absorption is no longer independent of the concentration, but it will be observed that for each of the three concentrations employed a different curve was obtained—B, B' and B'', respectively. Curve B for a concentration of 0.456 g. of phenolphthalein to the liter is fairly parallel to curve C, but with a definite decrease of intensity, while at a concentration of 0.114 g. to the liter, curve B', the absorption coefficient, has apparently

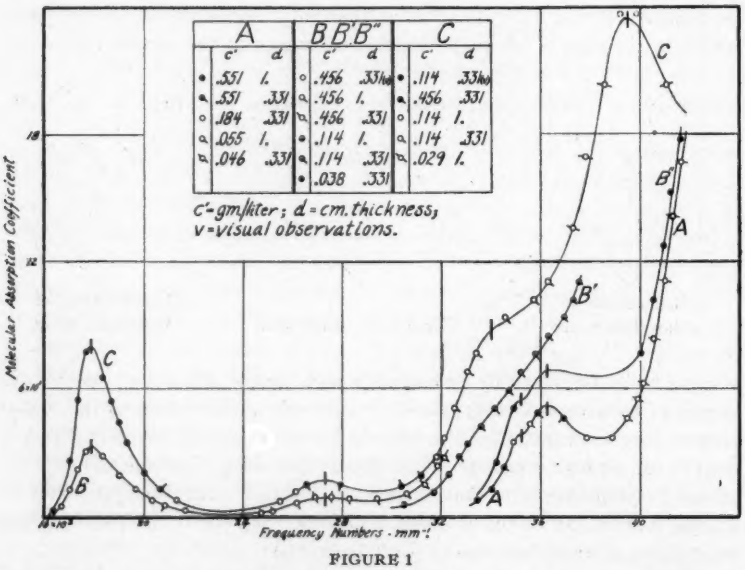


FIGURE 1

Phenolphthalein. A. Neutral absolute alcohol. B, B', B''. 1 molecule:23 molecules KOH. C. 1 molecule:232 molecules KOH.

decreased still further. Finally at 0.038 g. to the liter, curve B'', the absorption has reverted completely to the form found for the neutral solution, indicating that the vast majority of the molecules were then in the lactoid state with only a few present as ions, these latter contributing a pale pink color to the solution and being responsible for the somewhat greater intensity of the bands of curve B'' as compared with those of curve A. It may be pointed out that, had some particular concentration been chosen and the absorption studied by varying the thickness of the cell, a smooth curve would have been obtained since Lambert's law invariably holds, but such a curve would have been characteristic for that concentra-

tion and for the corresponding degree of hydrolysis only. The dubious value of this type of absorption data, unless amplified by data on other concentrations or for different ratios of alkali, is brought out below.

Results entirely analogous to the above have also been secured for compounds which are relatively strong acids compared to phenolphthalein, except that in these cases the hydrolysis is prevented when the excess of alkali is much smaller. The data on resorcinolbenzein, a mono-basic acid,

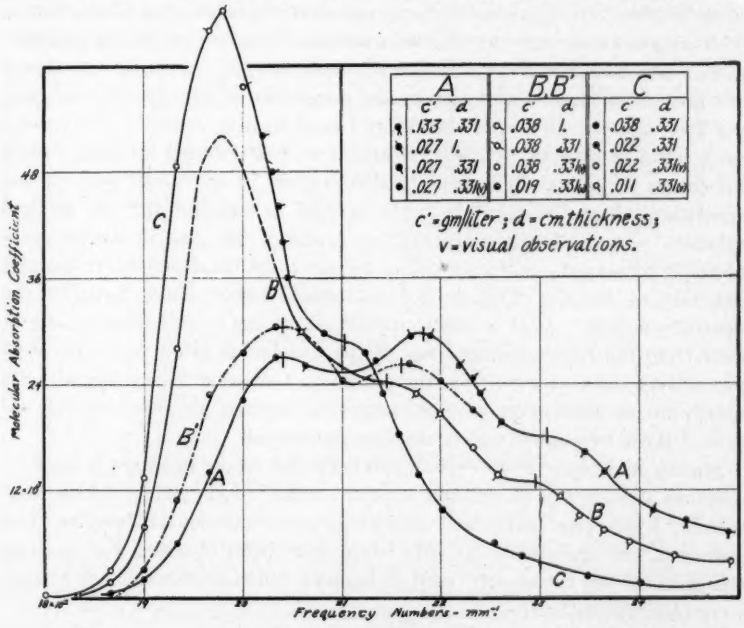


FIGURE 2

Resorcinol-benzein. *A*, Neutral absolute alcohol. *B*, *B'*, 1 molecule:1 molecule KOH; *C*, 1 molecule:2 molecules KOH.

and the parent substance of fluorescein, are typical of these stronger acids. Since resorcinolbenzein is colored in both the neutral and the alkaline solution, the changes may be readily followed in the visual region of the spectrum, as shown in figure 2, although the corresponding changes in the ultra-violet may not occur simultaneously as it has been found that the sensitivity of the various bands to change is not necessarily equal. This will be clear on examining figure 3, which presents all of the bands of resorcinol-benzein at the various ratios of alkali used. Curve *A*, figure 2, exhibits the absorption of the neutral solution, characterized by two bands of nearly equal intensity at frequency numbers 2040 and 2183. The

alkaline color, together with a bright green fluorescence, is well developed in the presence of two molecules of potassium hydroxide for each molecule of resorcinol-benzein, but this characteristic color is somewhat enhanced and the last trace of the ultra-violet band at 3624 in the neutral solution is effectively removed only in the presence of about twenty molecules of potassium hydroxide. The absorption of the alkaline solution in the visual region consists of a prominent band at 1980 and a shoulder at 2102, both of which are stable in the presence of two molecules of alkali over a wide range of concentrations of the resorcinol-benzein. With the addition of but one molecule of potassium hydroxide, however, two curves, B and B', are obtained corresponding to the concentrations 0.038 and 0.019 g. per liter, respectively. For the higher concentration, curve B, the prominent band is present at 1980, although with a lessened intensity, while indication of partial hydrolysis is also evident in the violet part of the spectrum, where the curve is fairly parallel to curve A for the neutral solution. At the lower concentration, however, the salt is almost completely hydrolyzed, for curve B' does not show the band at 1980 distinctly, but, instead, the pair of bands of almost equal intensity characteristic of the neutral solution. That a small amount of the ion is still present, is evident from the reversed intensities of the two bands of curve B' as compared with those of curve A: the presence of a trace of the ion with its maximum of absorption at 1980 serves to increase the intensity of the first of these two bands and to weaken the second.

Similar hydrolytic phenomena have been observed with other triphenylmethane derivatives of an acid nature: aurin,[7] fluorescein,[8] sulfonefluorescein,[9] hydroquinolphthalein[10] and hydroquinolsulfonephthalein[10] in alcoholic solution and, more recently, data have been obtained for aqueous solutions of the commonly used indicators, phenolsulfonephthalein and *o*-cresolsulfonephthalein.

The importance of such typical results as have been presented above for phenolphthalein and resorcinol-benzein lies primarily in the clue they give for the interpretation of spectroscopic data. It has frequently been the practice in the past, when determining the absorption spectra of compounds of the phthalein series, to use solutions of their "neutral" salts, mono- or dibasic as the case might be. Two reasons may be cited for this practice: the interest in colored solutions from the point of view of the limited chromophore theory and partly the insolubility of the original compound in water. As is now evident, however, such solutions do not yield the true absorption of the ion of the alkali salt but the composite absorption of the ion plus the product of hydrolysis, the neutral compound. The relative prominence of either of these will be determined by the hydrolytic constant of the salt and by the absolute concentration employed. As the first of these factors varies for the different substances, and as

previous investigators have frequently used different concentrations for each substance studied, it is difficult to find any sort of correlation in much of the published data for closely related compounds. The nature of this difficulty is brought out in figure 3, where the position of the absorption bands in the spectra of neutral and alkaline solutions of resorcinol-benzein, fluorescein and sulfonefluorescein has been indicated by vertical lines. The constitution of these three compounds is identical, except that fluorescein has a carboxyl and sulfonefluorescein a sulfoxyl group in place of an ortho hydrogen atom in the unsubstituted phenyl group of resorcinol-benzein. They are all relatively weak acids, resorcinol-benzein being monobasic, the other two dibasic, but in the order presented they become

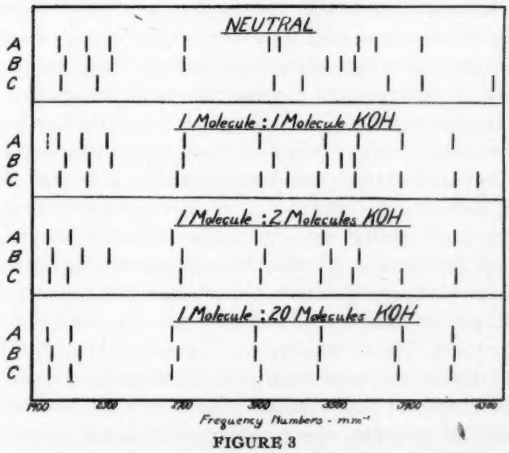


FIGURE 3

Positions of bands in alcoholic neutral and alkaline solutions of: *A*, Resorcinol-benzein; *B*, fluorescein; *C*, sulfonefluorescein.

progressively stronger, so that sulfonefluorescein should be least susceptible to hydrolytic change and this is actually found to be the case. The first horizontal row of figure 3 gives the bands for the neutral alcoholic solutions. Because of differences in internal structure of the free compounds, such as the formation of an inner salt with sulfonefluorescein, there is very little correlation among the three sets of bands, except for the first four bands of resorcinol-benzein and fluorescein. In the presence of one and two molecules of potassium hydroxide, as shown in the second and third rows, respectively, it is evident that there are bands present characteristic of both the free compound and the ion and that no regular relationship between the disappearance of the former and the appearance of the latter

exists for the three compounds. However, with the addition of twenty molecules of alkali all of the neutral bands have vanished and the fourth horizontal row exhibits a perfect accord for the positions of the absorption bands due to the ions of these compounds, which is equally well shown by the relative intensities of the bands on the original absorption curves.[6,8,9] It is now possible to conclude on the basis of such data that the ions of resorcinol-benzein, fluorescein and sulfonefluorescein have entirely similar structures and that the differences found in the neutral solutions must be ascribed to variations in internal structure.

It is to be expected that this phenomenon of hydrolysis would also appear as a complicating factor in the study of the fluorescence of compounds such as fluorescein, which require the addition of alkali to bring out the maximum luminescence. Since the fluorescent radiation suffers absorption by the solution as it passes through, corrections are necessary in order to obtain the true spectral distribution of the emitted radiation. If the amount of alkali present is small, as would be the case if the commercial disodium salt of fluorescein, uranine, were employed, it is obvious that absorption measurements must be made on the solution actually used for the determination of the luminescence and not on one of a different, perhaps more convenient, concentration. There also exists the possibility, although more experimental data are needed to settle this point, that the fluorescence of the neutral solution has a different spectral distribution from that of the alkaline solution. Under these circumstances, of course, if the conditions are favorable for hydrolysis the observed fluorescence will be a composite of the two forms. In any event, in studying such fluorescent media the safest procedure would be to have a sufficient excess of alkali present so that no question could arise as to hydrolysis entering in, particularly at the low concentrations frequently required for such measurements. The amount necessary in any given case can readily be determined by preliminary observations on the absorption of the solution at several concentrations, as outlined above.

[1] The investigations upon which this article is based were supported by grants from the Heckscher Foundation for the Advancement of Research, established by August Heckscher at Cornell University.

[2] A portion of the material here presented was read at the meeting of the American Physical Society, Washington, D. C., April, 1928.

[3] Heckscher Research Assistant in Physics and Chemistry, Cornell University, 1923–1928.

[4] Speas, *Physic. Rev.*, **31**, 569 (1928); Holmes, *Ind. Eng. Chem.*, **16**, 35 (1924).

[5] Orndorff, Gibbs and McNulty, *J. Amer. Chem. Soc.*, **48**, 1327 (1926).

[6] Orndorff, Gibbs and Shapiro, *Ibid.*, **48**, 1994 (1926).

[7] Orndorff, Gibbs, McNulty and Shapiro, *Ibid.*, **49**, 1545 (1927).

[8] Orndorff, Gibbs and Shapiro, *Ibid.*, **50**, 819 (1928).

[9] Gibbs and Shapiro, *Ibid.*, **50**, 1755 (1928).

[10] Shapiro, *Ibid.*, **50**, 1772 (1928).

FURTHER REMARKS ON THE SECOND LAW OF THERMODYNAMICS IN GENERAL RELATIVITY

BY RICHARD C. TOLMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Introduction.*—In a previous article[1] I have attempted an extension of thermodynamics to general relativity, and in following articles[2] have discussed certain applications of this extension. The postulate that was taken in this work as the analogue of the ordinary first law of thermodynamics was the same as the modified principle of the conservation of energy given by Einstein[3] as applying in the case of general relativity considerations. And in spite of the early criticisms of Einstein's principle, based on the fact that it could not be expressed in the form of a tensor equation, I feel that Einstein's treatment is now safely regarded as satisfactory, since the principle *is* expressible by an equation which is true for all sets of coördinates. The postulate, which was taken in the work as the analogue of the second law of thermodynamics, was however a new one and the main purpose of the present article is to present some further reasons which led me to the principle chosen.

In the previous article the new postulate was stated in the form which it takes when applied to an isolated finite system, and was justified by showing that it reduced to the ordinary second law in the limiting case of flat space-time, and by showing that it agreed with the principle of covariance because of its expression in tensor form. In the present article, however, the new postulate will be presented in the form which it takes when applied to a non-isolated infinitesimal system, and will be justified by showing that it can be regarded as a very natural result of generalizing the older thermodynamics. To do this we shall first obtain from the older thermodynamics an expression which embodies the results of the second law as applied to an infinitesimal four-dimensional region in flat space-time. Proceeding on the basis of the equivalence hypothesis we shall then regard this expression as true for an infinitesimal region even in curved space-time. And, finally, we shall generalize so as to put this expression in covariant form, and thus obtain the desired modification of the second law.

It will also be shown as a further justification that the new form of the second law leads to an expression for the entropy of a system in a stationary state, which agrees with what is to be expected on the basis of the usual relation between entropy and probability. And the article will in addition present an opportunity to make a number of incidental remarks which may be of a clarifying nature.

2. *Application of Ordinary Thermodynamics to an Infinitesimal Region.*—Let us first examine the application of the classical thermodynamics to an infinitesimal region in flat space-time, employing Galilean coördinates, x, y, z and t, corresponding to the line element

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2. \tag{1}$$

If we consider the material contained in the infinitesimal spatial volume $\delta x \delta y \delta z$, it would seem to be the essence of the second law that the increase in the entropy of this region which occurs in a time interval δt should be equal to or greater than the entropy which is brought into the region during that interval by the actual convection of material or by the flow of heat. Hence, we may write as an expression of the requirements of the second law, the inequality,

$$\left(\frac{\partial \phi}{\partial t}\right) \delta x \delta y \delta z \delta t \geqq -\left\{\frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial y}(\phi v) + \frac{\partial}{\partial z}(\phi w)\right\} \delta x \delta y \delta z \delta t + \frac{\delta Q}{T} \tag{2}$$

where ϕ is the density of entropy, u, v and w are the velocities of macroscopic flow at the point in question, and $\delta Q/T$ is the quotient of the heat entering the region in the time δt, divided by the temperature of the boundary.[4]

In accordance with the special theory of relativity, however, which holds in the limiting case of flat space-time, we may now transform equation (2) in the following manner. Since entropy is an invariant for the Lorentz transformation, entropy density will evidently be affected by the Lorentz-Fitzgerald factor of contraction ds/dt, in such a way that we may make the substitution

$$\phi = \phi_0 \frac{dt}{ds} \tag{3}$$

where ϕ_0 is the proper density of entropy referred to coördinates stationary with respect to the macroscopic velocity of the material, and ds refers to the macroscopic motion of the material. Furthermore, since heat and temperature have the same transformation factors, on the basis of the special theory, we may also introduce the substitution

$$\frac{\delta Q}{T} = \frac{\delta Q_0}{T_0}. \tag{4}$$

Making these substitutions, transposing terms, and writing explicit expressions for the components of velocity u, v and w, we may now rewrite (2) in the form,

$$\left\{\frac{\partial}{\partial x}\left(\phi_0\frac{dt}{ds}\frac{dx}{dt}\right)+\frac{\partial}{\partial y}\left(\phi_0\frac{dt}{ds}\frac{dy}{dt}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dt}{ds}\frac{dz}{dt}\right)+\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)\right\}\delta x\delta y\delta z\delta t \geqq \frac{\delta Q_0}{T_0} \quad (5)$$

and by cancellation, we obtain the symmetrical expression,

$$\left\{\frac{\partial}{\partial x}\left(\phi_0\frac{dx}{ds}\right)+\frac{\partial}{\partial y}\left(\phi_0\frac{dy}{ds}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dz}{ds}\right)+\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)\right\}\delta x\delta y\delta z\delta t \geqq \frac{\delta Q_0}{T_0} \quad (6)$$

3. *Generalization to Curved Space-Time and Curvilinear Coördinates.*—Expression (6) has been shown to be true in the limiting case of flat space-time using Galilean coördinates. Since it applies to an infinitesimal region in space-time, we shall now postulate, on the basis of the equivalence hypothesis, that it is also true for the case of curved space-time provided we use natural coördinates, and is a special case of a more general covariant expression true in all sets of coördinates.

A simple covariant expression of which (6) is a special case may now be obtained as follows. Let us define the *entropy vector* at the world-point in question by the equation

$$S^\mu = \phi_0\frac{dx_\mu}{ds} \quad (7)$$

and the corresponding *entropy vector density* by the equation

$$\mathfrak{S}^\mu = S^\mu\sqrt{-g} = \phi_0\frac{dx_\mu}{ds}\sqrt{-g}. \quad (8)$$

The desired generalized covariant expression may now be written in the form

$$\frac{\partial\mathfrak{S}^\mu}{\partial x_\mu}\delta x_1\delta x_2\delta x_3\delta x_4 \geqq \frac{\delta Q_0}{T_0}. \quad (9)$$

To show that this expression is generally covariant, we note that it can be written, by a well-known transformation,[5] in the form

$$(S^\mu)_\mu\sqrt{-g}\ \delta x_1\delta x_2\delta x_3\delta x_4 \geqq \frac{\delta Q_0}{T_0} \quad (10)$$

where $(S^\mu)_\mu$ is the contracted covariant derivative or divergence of S^μ. The quantities $(S^\mu)_\mu$ and $\sqrt{-g}\ \delta x_1\delta x_2\delta x_3\delta x_4$ are however known to be invariants, while $\delta Q_0/T_0$ is obviously an invariant, so that both sides

of expression (10) are tensors of rank zero, and the requirement of covariance is simply met.

To show that expression (9) reduces in the limit to the original expression (6), we note that in flat space-time using Galilean coördinates the quantity $\sqrt{-g}$ is a constant equal to unity. Substituting this value for $\sqrt{-g}$ in (8) and substituting the result into (9), we then easily obtain the earlier expression (6).

Hence the simple expression (9) is a natural covariant generalization of the requirements of the second law in flat space-time, and we shall assume it to be a correct statement of the general form of the second law.[6]

4. *Application to a Finite Isolated System.*—If we now integrate (9) over the whole of an isolated system and over any desired time interval, the summation of $\delta Q_0/T_0$ over the interior of the system will cancel out, since any heat entering a given element of volume is abstracted from neighboring elements. Hence for an isolated system we shall obtain the result

$$\int\int\int\int \frac{\partial \mathfrak{S}^\mu}{\partial x_\mu}\, dx_1 dx_2 dx_3 dx_4 \geqq 0. \tag{11}$$

And this is the expression which was taken in the earlier work as a statement of the second law.

5. *Entropy of a System in a Stationary State.*—Let us now consider a system having the property that we can find some set of coördinates x_1, x_2, x_3, x_4 such that the velocities corresponding to the space-like coordinates will be everywhere equal to zero, in accordance with the equations

$$\frac{dx_1}{ds} = \frac{dx_2}{ds} = \frac{dx_3}{ds} = 0. \tag{12}$$

For convenience we shall speak of such a system as being in a stationary state.

With these values for the spatial velocities, it is evident, from the equation of definition (8) for the vector density $\mathfrak{S}^\mu$, that the general expression (11) reduces to

$$\int\int\int\int \frac{\partial}{\partial x_4}\left(\phi_0 \frac{dx_4}{ds}\sqrt{-g}\right) dx_1 dx_2 dx_3 dx_4 \geqq 0 \tag{13}$$

or, performing the indicated integration with respect to the time-like coördinates over an interval x_4 to x_4', we obtain

$$\left|\int\int\int \phi_0 \frac{dx_4}{ds}\sqrt{-g}\, dx_1 dx_2 dx_3\right|_{x_4}^{x_4'} \geqq 0. \tag{14}$$

And this expression can be conveniently transformed by using the general relation[7]

$$\sqrt{-g}\, dx_1 dx_2 dx_3 dx_4 = dV_0 ds \tag{15}$$

to give us

$$\left| \int \phi_0 dV_0 \right|_{x_4}^{x_4'} \geqq 0 \tag{16}$$

where dV_0 is the element of *proper* spatial volume, and the integration is to be taken over the whole of the isolated system.

We have thus obtained for such a system a simple quantity $\int \phi_0 dV_0$ whose value can only increase with the "time" x_4, and which hence might be called the entropy of the system. This quantity, however, is seen to be the summation, over the whole system, of the ordinary entropies of its parts, and by introducing for each of these parts the ordinary relation between entropy and probability

$$S = k \log W \tag{17}$$

and taking the probability of the arrangement of the total system as the product of the probabilities for each of its parts, we see that equation (16) also implies that the system could change with increasing values of x_4 only to arrangements of greater probability.

Thus, for a system "in a stationary state" we have obtained an additional justification for our generalized form of the second law.

6. *Incidental Remarks.*—In conclusion certain incidental remarks may be of interest.

In connection with the process of generalization by which we have passed from equation (6), valid in the special theory of relativity, to equation (9), valid in the general theory, it should be emphasized that equation (9) is in no sense strictly deduced from (6). Equation (9) may be regarded as a postulate from which (6) can be deduced as a special case, but not vice versa. It is of the essence of a real generalization that it should contain something not present in the special case which suggested it. Hence our postulate, equation (9), has not been proved necessarily true, but if true adds something genuinely new to theoretical physics.

With regard to the application of the second law to a finite isolated system as given in section 4, it should be noted that it is assumed as one of the characteristics of an isolated system, that no heat should cross the external boundary as measured by observers stationary with respect to that boundary.

In connection with the distinction between space-like and time-like coördinates made in section 5, and often arising in general relativity considerations, it is well to remember that the space-time manifold is best regarded not as four dimensional but rather as three-plus-one dimensional. In the case of the actual problems that have arisen in the field of general

relativity, an examination of the line element involved always shows that one of the coördinates, say x_4, is distinguished by the fact that the corresponding component of the fundamental tensor, g_{44}, is always positive. This coördinate is the time-like one and the others space-like.

[1] Tolman, R. C., these PROCEEDINGS, **14** (1928), 268–72.

[2] *Ibid.*, **14** (1928), 348–53, and **14** (1928), 353–6.

[3] Einstein, A., *Berl. Ber.*, **1918**, 448–59.

[4] It should be noted that what follows would not be affected if we took different temperatures for different parts of the boundary, and took $\delta Q/T$ as symbolic of a summation of such quantities for the different parts of the boundary.

[5] See Eddington, A. S., *The Mathematical Theory of Relativity*, Cambridge, **1923**, equation (51.12).

[6] Equation (9) was presented by the author to the Astronomy Physics Club of Pasadena on March 2, 1928. It should be compared with equation (11) of M. Tu. De Donder, *Comptes Rend. Séance*, du **11**, Juin, 1928, p. 1601.

[7] See Eddington, loc. cit., equation (49.42).

SERIES SPECTRA OF CADMIUM-LIKE ATOMS

BY J. B. GREEN AND R. J. LANG

MENDENHALL LABORATORY, OHIO STATE UNIVERSITY AND PHYSICAL LABORATORY, UNIVERSITY OF ALBERTA

Communicated July 23, 1928

The classification of spectra, homologous with that of cadmium, previously investigated by Green and Loring[1] and by Lang[2] has now been extended to Sb IV. The plates were taken and measured at the University of Alberta. A high potential spark in vacuum was used as a source.

The spectrum of Sb IV was classified in the usual way, by plotting the $\sqrt{\nu/R}$ values of the terms of Cd I, In II, and Sn III and extrapolating the curves thus found for each of the terms to Sb IV (a Moseley diagram). This is shown in figure 1. In this way we were enabled to find the approximate values of the terms of Sb IV and thus the approximate positions of the principal lines. In addition to this, the frequencies of lines corresponding to jumps between terms with the same principal quantum number increases uniformly in going from one element to the next. This corresponds to an assumption of the irregular doublet law of x-ray spectra, and is shown clearly in table 1.

Having determined the approximate positions of the groups, it was necessary to determine the multiplet separations. This was done by aid of the fourth power law (regular doublet law), assuming that this law

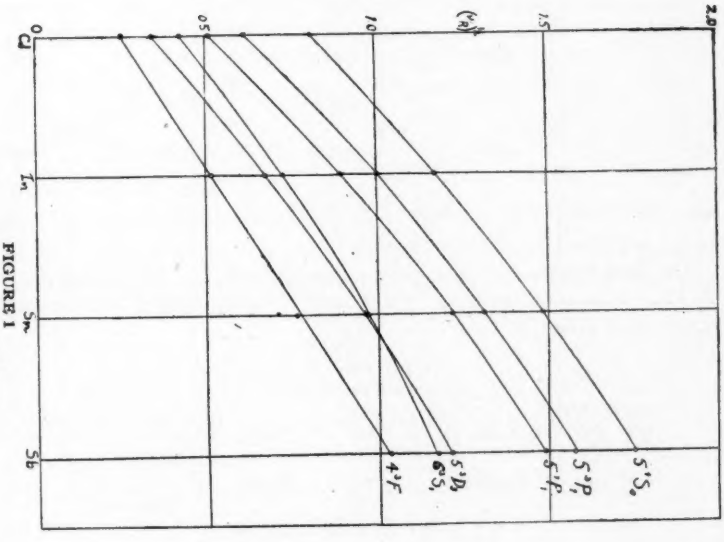

FIGURE 1
Term values of Cd I-Sb IV.

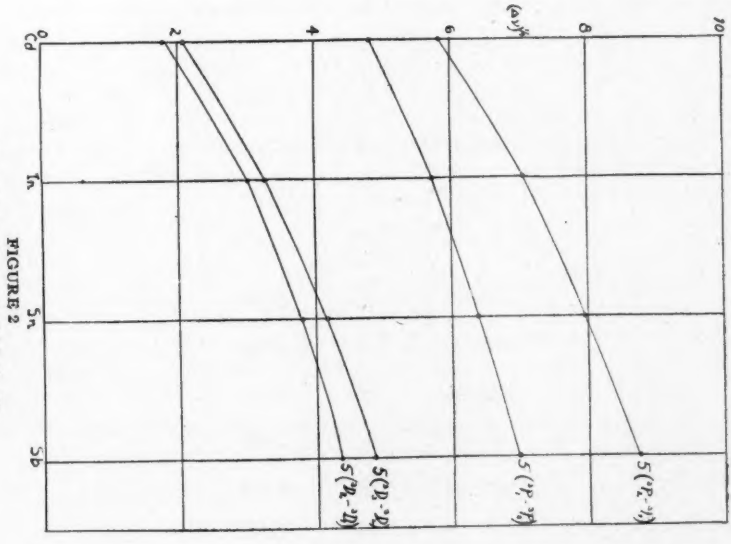

FIGURE 2
Regular doublets of Cd I-Sb IV.

TABLE 1

IRREGULAR DOUBLETS

ELEMENT	5^3P_2—5^3D_3	5^3P_1—$5^3P_1'$	5^1S_0—5^1P_1	5^1S_0—5^3P_1
Cd I	27689	44088	43692	30656
	28789	15810	19345	11871
In II	56478	59898	63037	42527
	26129	15023	16871	12664
Sn III	82607	74921	79908	55191
	24648	14764	16052	10887
Sb IV	107155	89685	95960	66078

also holds for triplet spectra. The results are shown in figure 2 where $(\Delta\nu)^{1/4}$ is plotted against atomic number.

The final results for Sb IV are given in table 2. The lowest level of Sb IV, 5^1S, is approximately 340,000 cm.$^{-1}$, corresponding to an ionization potential of about 42 volts.

TABLE 2

THE SPECTRUM OF SB IV

CLASSIFICATION	λ	INT.	ν	$\Delta\nu$	
5^3P—6^3S	861.61	(15)	116062		
				5851	
	820.26	(15)	121913		
				2260	
	805.33	(12)	124173		
5^3P—5^3D	940.22	(5)	106358		
				355	
	937.09	(30)	106713		
				553	
	932.26	(30)	107255		
					5854
	891.17	(18)	112212		
				352	
	888.38	(20)	112564		
					2261
	873.57	(15)	114473		
5^3D—4^3F	.....	..	...		
	2113.14	(3)	47323		
				160	
	2106.00	(10)	47483		
					557
	2092.26	(2)	47795		
				95	
	2088.53	(10)	47880		
					351
	2076.99	(5)	48146		
5^3P—$5^3P'$	1192.94	(35)	83827		
				5858	
	1115.01	(15)	89685		
				2262	

	1087.58	(15)	91947	
	1120.43	(10)	89251	
				5866
	1051.33	(10)	95117	
	1145.86	(15)	87271	
5^1S—5^3P_1	1513.34	(15)	66078	
5^1S—5^1P	1042.10	(40)	95960	
5^1P—6^3S	1086.50	(5)	92039	
		Calc.	92031	
5^1P—5^3D_1	1214.78	(5)	82319	
		Calc.	82326	

Returning now to In II, it has been possible to extend the investigations to the determination of second members of series. The results are given in table 3.

TABLE 3

SPECTRUM OF IN III*

CLASSIFICATION	λ	INT.	ν	Δν	
5^3D—5^3F	3160.7		31629		
				111	
	3149.6		31741		
	.....		...		
6^3S—6^3P	6891.28	(3*u*)	14507		
				558	
	7167.88	(2*v*)	13949		
				182	
	7262.98	(1*u*)	13767		
5^3P—7^3S	1305.58	(0)	76594		
				2477	
	1264.69	(00*d*)	79071		
	.....	..	...		
5^3P—6^3D	.....	..	...		
	1290.43	(1)	77493		
				38	
	1289.80	(2)	77531		
					2479
	1250.98	(00)	79936		
				36	
	1250.43	(0)	79972		
	.....	..	...		
5^1S—5^3P_1	2350.73	(2)	43527		
5^1P—6^3S	3325.44	(2)	30064		
5^1P—5^3D_2	2610.8	(1)	38291		
5^1P—5^1D	2492.7		40105		
5^3P_1—5^1D	1649.95	(5)	60608		
		Calc.	60615 (may be Pb)		
5^1D—4^1F	4847.85		20622		
5^3F—5^3G	6198.06	(4)	16130		

It is hoped later to extend the investigation to include the spectrum of Te V and second members of series of Sn III.

* Wave-lengths are taken from Schulemann, *Zeit. Wiss. Phot.*, **X**, 263 (1911). Intensities whenever listed. The triplet 5^3D—4^3F was given in reference 1.

[1] Green and Loring, *Phys. Rev.*, **30**, 585 (1927).

[2] Lang, *Ibid.*, **30**, 769 (1927).

THE HYPOSTASE: ITS PRESENCE AND FUNCTION IN THE OVULE OF THE ONAGRACEAE

BY DONALD A. JOHANSEN*

DEPARTMENT OF BOTANY, STANFORD UNIVERSITY

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In a recent preliminary note[1] I indicated that earlier theories regarding the presence of the hypostase and epistase in the ovule of the Onagraceae, and particularly those hypotheses advanced concerning the function, or functions of the former, were in need of revision. Since that note was written (July, 1927), cytological and ecological observations have been made on 52 species and varieties whose natural habitats represent all environmental gradations from extreme xerophily to hydrophily. It is believed that the statistics accumulated as a result of this study, together with a reinterpretation of the observations of other workers, permit the formulation of an entirely new set of conclusions supported, in some instances, by experimental evidence.

The hypostase was first described by Ph. van Tieghem[2] in 1893, and the subject was further elaborated by the same author in two succeeding papers.[3,4] The scanty work done in other families may be disregarded and attention confined to the Onagraceae. Of the vast number of writers upon this family, only five deal with the hypostase, and in each case as a side issue. These investigators, Täckholm,[5] Werner,[6] Ishikawa[7] and van Wisselingh,[8] all base their observations upon prepared material alone.

Essentially, as seen in preparations made by orthodox cytological procedures, the hypostase consists of a well-defined but irregularly outlined group of thick-walled cells at the chalazal end of the anatropous ovule, situated between the bases of the two integuments and directly on top of the end of the vascular bundle entering from the raphe, and which ordinarily but not invariably extends into the nucellus almost to the base of the megagametophyte. It is connected more or less directly with the inner integument, but is not an integumentary structure. The cells are thick-walled, heavily cutinized, and are filled with some substance or a

mixture of substances which stains so intensely with the usual chromatin stains that attention is at once attracted when a preparation is examined under the microscope. Ishikawa is doubtless mistaken in referring to the hypostase as "chromatin-containing." Werner states that the cells lack nuclei, but both Ishikawa and the writer have always been able to locate a small nucleus flattened against one wall. In certain species, e.g., *Clarkia elegans*, a secondary group of cells similar to the hypostase, in general characteristics and known as the epistase, is often found between the integuments at the apex of the nucellus in maturing seeds, and in other species, e.g., *Zauschneria californica*, the epistase is present and in addition the chalazal hypostase is separated into two equal tiers by the interposition of a layer of empty cells. The cupule form of the hypostase, as originally described by van Tieghem, is rarely found, appearing at its best in *Hartmannia tetraptera*. The hypostase of *Epilobium watsoni* var. *franciscanum* is peculiar in that it extends into the raphe as far as the funiculus. Other unusual forms will probably be found as more species are examined.

Täckholm and Werner conclude that the hypostase is concerned in the nutrition of the megagametophyte; van Wisselingh, that it is merely a cork tissue; while Ishikawa denies that the hypostase has any nutritive function and concludes that it is simply a depository for some "ultimate substance." It soon became apparent that those writers who assign to the hypostase the principal rôle in the nutrition of the megagametophyte and embryo failed to take into account the absence of this structure in many species. Moreover, there is one important consideration which seems to have been overlooked by these same investigators, namely, that hypostase cells possess neither the structure nor appearance of absorbing organs—a prerequisite if their theory is to be considered tenable. Other nullifying evidence comprises the very thick cell walls and the inconspicuous flattened nuclei. It was consequently felt that a new explanation of the presence of the hypostase in certain species, as well as of its absence in others, was required.

Of the 52 species noted in the first paragraph, 18 lack a hypostase. Comparison of the data regarding the presence or absence of the hypostase with the known habitat of the plant in nature[9] revealed that species growing in the water (*Ludwigia mulertti*), in boggy situations (*Circaea pacifica*), in damp places or near waterfalls (*Epilobium obcordatum*), in places where there is sufficient moisture present in the soil to keep it from drying out completely during late summer (*Epilobium minutum*), or have the ovules enclosed in fleshy berries (most species of *Fuchsia*), lack completely all evidence of a hypostase. On the other hand, species inhabiting regions in which a long summer drought prevails (e.g., the Mojave Desert and lower San Joaquin valley in California), invariably

possess well-developed hypostases and, in a few cases, an epistase. Intergrades occur; for example, *Godetia amoena* lacks a hypostase, but the variety *lindleyi*, which is usually found in drier situations, possesses a very small and flattened one. Many species, e.g., *Anogra trichocalyx* and *Epilobium paniculatum*, show lack of regularity in its appearance, the evidence showing that this condition is determined by the amount of available soil moisture. The hypostase generally makes its appearance during the progress of meiosis in the ovule; but in *Boisduvalia glabella* var. *campestris* it appears rather late, and in *Sphaerostigma dentata*, *S. veitchiana* and *Galpinsia hartwegi*, it does not appear until after embryo development is well under way. The epistase, when present, does not appear until after cotyledon development in the embryo has begun. It is found only in species inhabiting arid situations, in those which bloom rather late (e.g., *Zauschneria latifolia*), and in cultivated species.

In experiments performed on potted plants of species normally lacking a hypostase, the soil was suddenly allowed to become quite dry. Capsules maturing during this period contained ovules possessing well-defined hypostases. Seed was also saved from these plants and grown the following season under garden conditions; the ovules of the progeny did not show the presence of hypostases. The evidence seems sufficient to demonstrate that the hypostase is an acquired character.

In stained preparations all degrees of reaction to the same stain (safranin), from none at all to the highest possible intensity, occur. It is apparent that the density of the fluid or substance contained in the cells of the hypostase determines the degree of precipitation or coagulation by killing fluids and consequently of staining reaction. The density of the fluid is in turn determined by the amount of available soil moisture; that is to say, the less the moisture available, the greater the density of the fluid.

Stated briefly, the writer concludes from the evidence at hand that the hypostase is a flexible adaptation to the environment; it makes its appearance only when necessity requires it. To account for the function of the hypostase and epistase when either or both are present, it may be said that they serve to stabilize the water balance of the resting seed over the long period of dormancy during the hot dry season.[10] There is some slight but inconclusive evidence that in certain species the hypostase plays, during the germination of the seed, a rôle analogous to that of the so-called "aleurone layer" found in the seeds of the Gramineae.

* NATIONAL RESEARCH FELLOW IN THE BIOLOGICAL SCIENCES.

[1] Johansen, *Madroño*, **1**, 165–167 (1928).

[2] van Tieghem, *Bull. de la soc. bot.*, **40**, 347 (1893).

[3] van Tieghem, *Bull. du Mus. d'Hist. Nat., Paris*, **7**, 412–418 (1901).

[4] van Tieghem, *Ann. sci. nat., bot. VIII*, **17**, 347–362 (1903).

[5] Täckholm, *Svensk bot. Tidsskr.*, **8**, 223–234 (1914).

[6] Werner, *Beih. z. bot. Centralbl.*, **32**, 1–14 (1915).

[7] Ishikawa, *Ann. Bot.*, **32**, 279–317 (1918).

[8] van Wisselingh, *Pharm. Weekblad*, **57**, 77–83, 125–129 (1920).

[9] In the great majority of the species examined, the material was collected directly from plants growing in their natural habitat, and only a few came from the garden. The latter were mostly species long in cultivation.

[10] I am greatly indebted to Dr. George J. Peirce, who first ventured the suggestion that the hypostase might play some part in the preservation of the water balance of the resting seed.

GLYCOGEN AS A MEANS OF CILIARY REVERSAL

BY G. H. PARKER

ZOÖLOGICAL LABORATORY, HARVARD UNIVERSITY

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Many years ago I called attention to the fact that the cilia that cover the lips of the common sea-anemone, Metridium marginatum, and that ordinarily beat outward from the mouth, reverse the direction of their stroke and beat inward when they are bathed with a seawater extract from such food materials as crab-meat or mussel-meat (Parker, 1905*a*).[1] By means of this reversal the sea-anemone is enabled to appropriate from the surrounding seawater bits of material serviceable as food. But not only do meat juices bring about this reversal; other substances such as Witte's peptone, deuteroalbumose, asparaginic acid, creatine, and even potassium chloride, also call forth this reaction (Parker, 1905*b*).[2]

During the past season at the Marine Biological Laboratory at Woods Hole, I had occasion to review some of these reactions and opportunity was offered to test the effects of glycogen on these cilia. A fully expanded sea-anemone in a shallow dish of seawater was tested in the following way. Small squares of clean filter paper soaked in seawater were placed on the ridged lips of this animal and their direction of movement was recorded. They commonly moved outward over the lips to the bases of the innermost tentacles where they were taken up by the tentacular cilia and discharged from the disk of the sea-anemone. Before they were discharged, however, they could be picked up by fine forceps and in their wet condition thoroughly worked full of finely powdered glycogen. This material in seawater became semi-gelatinous. When the piece of filter paper thus permeated with jelly-like glycogen was replaced on the lip of the sea-anemone at precisely the spot from which it formerly had been swept outward, it was found to move with as great regularity inward toward the mouth. This response was uniformly and persistently carried out and is believed to be due, for reasons already given (Parker and Marks,

1928),[3] to the reversal of the ciliary stroke on the lips of Metridium. In consequence of the relative abundance of glycogen in the meat of mussels such a response might have been anticipated. No reversal was obtained from ordinary starch or from glucose when these substances were applied to the lips of the sea-anemone as the glycogen was. I conclude that glycogen may be added to the short list of substances by which ciliary reversal in sea-anemones may be induced.

[1] Parker, G. H., 1905*a*. "The Reversal of Ciliary Movement in Metazoans," *Amer. J. Physiol.*, **13**, 1.

[2] Parker, G. H., 1905*b*. "The Reversal of the Effective Stroke of the Labial Cilia of Sea-Anemones by Organic Substances," *Amer. J. Physiol.*, **14**, 1.

[3] Parker, G. H., and A. P. Marks, 1928. "Ciliary Reversal in the Sea-Anemone Metridium," in press.

THE PRODUCTION OF MUTATIONS BY X-RAYS[1]

BY H. J. MULLER

DEPARTMENT OF ZOÖLOGY, UNIVERSITY OF TEXAS[2]

Read before the Academy April 24, 1928

1. *Earlier Experiments—Magnitude of the Effect.*—In the earlier experiments on this subject, which may first be referred to briefly, crosses were made between flies containing a different collection of genes in their X-chromosomes (the males having bobbed, and the females scute, vermilion, forked). In most cases only one parent was subjected to the X-ray treatment. The female offspring from these crosses would contain any new mutant gene that might have arisen in only one of their X-chromosomes, and hence they would manifest no abnormality if, as usual, the mutant gene were recessive. By breeding them, however (they were bred to their brothers), males would be produced in the next generation that manifested any visible mutation occurring in their X, or, if the mutation were lethal, certain expected classes of males would then fail to appear. The counts would also show in which chromosome—that from the original male or female—the mutation lay, and at approximately what locus in that chromosome it lay. And since, owing to the nature of the cross, some flies of the same composition as those of the first offspring generation were again produced in the second offspring generation, the same cross could then be repeated, and repeated again, in order to determine whether further mutations continued to originate in later generations, long after treatment. Slightly over a thousand cultures of the flies ("F_1") whose parents had been treated with varying doses of x-rays (called *t*1, *t*2, etc.) were started simultaneously, and nearly the same number of controls.

When the offspring ("F_2") from the above cultures began to be examined (November, 1926) a decided difference between the control and treated series was soon evident. The controls showed a very low frequency of lethal mutations (1 in 947 fertile cultures), like that usually found, whereas the treated series showed a surprisingly high frequency of lethals (88 in 758 cultures), and all but 3 of these lethals were confined to the chromosomes derived from the treated progenitor. They occurred in abundance, moreover, no matter whether the treated progenitor had been a female fly or a male. Similar results were found with regard to visible mutations and semi-lethals in these cultures, though of course they were not as numerous as the lethals. In later generations ("F_3" and "F_4"), only a small number of new mutations were found, not significantly more in the treated than in the control series.

A new series of crosses was then initiated (spring of 1927), similar in principle to the preceding, but in which the X-chromosome introduced from the untreated female progenitor contained the grouping of genes called "C_l," which prevents crossing over, and which is lethal in its effect on males. If now the offspring females, which were subjected to the breeding test, had also received a new lethal, in the X-chromosome from their treated father, they could not produce sons containing this X-chromosome, or sons containing the C_l chromosome either, that is, they could produce no sons at all—a condition definitely determinable by naked-eye inspection of their culture bottles. Somewhat over 1600 (fertile) cultures were examined in this set of experiments, and the results were substantially the same as before. In both sets of experiments, moreover, the treated progenitors were bred through two broods, and the offspring from the second brood, from germ-cells effecting fertilization a week or more after treatment, were found to have sensibly the same frequency of mutant genes as the others. On adding together the results of both sets of experiments, the lethal mutation frequency was found to be of the order of 150 times higher in the heavily treated *t*4 series than in the controls, and also significantly higher in the *t*4 than in the half as heavily treated *t*2 lots—something of the magnitude of one and a half times to twice as high, though the precise factor here is as yet uncertain.

To obtain more abundant data on visible mutations, rather than the ubiquitous lethals, crosses were made at the same time of treated males to females having attached X-chromosomes and a Y. Here the sons carry their mother's Y and their father's X and hence reveal on inspection all visible mutations, even recessives, that arose in this X of the treated sperm. Counts showed that the abnormalities thus produced were very abundant, and similar results have been obtained later on repetitions of this experiment.

Combining all the work done in 1926 and 1927, the following sex-linked

mutations had been induced (one or more times) that appeared allelomorphic or identical with previously known mutations: broad, white, eosin, facet, Notch, crossveinless, tan, vermilion, tiny bristles, miniature, garnet or ruby, small wing, rudimentary, uneven eye, forked, small eye, cleftoid and bobbed. Of these, at least one each of the following had been proved, by crossings with the old mutants, to be really allelomorphic to them: white, Notch, tan, vermilion, small wing, small eye, forked and bobbed.

The data on induced mutations in the X were sufficient to allow a graph to be made, showing their relative frequencies throughout the length of the "map." The graph disclosed that they were heaped more abundantly in the same regions as those in which more spontaneous mutations had previously been found. This unevenness is probably due merely to variations in our measure of length—the cross-over frequency—from region to region of the chromosomes.

Though most of the studies to date have been directed especially at the X-chromosome, various autosomal mutations also have been found, both dominants and recessives. In one experiment they were looked for at six specific loci (in a cross of x-rayed wild-type males by females homozygous for roughoid, hairy, scarlet, pink, spineless, ebony) and found at two of them (scarlet and spineless), though in each case combined with lethal effects that might suggest "deficiencies." Among the dominant autosomal mutations which have been induced are star eye, hairless, mutations resembling in appearance delta and truncate, and many of the minute bristle type; also dominant mutant allelomorphs of vestigial, and of eyeless (see below). The data on dominants indicate that mutations are induced as readily in the autosomal chromatin as in that of the X-chromosome.

The above lists include only those induced mutations which seemed the same as, or allelomorphic to, mutations already familiar to the Drosophila worker. There were also diverse new types of mutations produced, both in the X and in the other chromosomes, but space forbids their being passed in review here.

2. *Possible Nature of the Effect and Structure of the Gene.*—A considerable number of the induced mutations have been carried through a protracted series of breedings, and in the great majority of cases—with one exception, to be noted later—they breed quite true, as stable Mendelian genes. Their stability is in fact evident almost from the moment of their origination, for if the mutant gene mutated again, backwards or forwards, in the cells of the early embryo derived from the mutant germ cell, the resulting adult would show an irregular patchwork of mutant tissues, scattered among normal tissues. This is not the case. It can be shown however that, on the theory that the gene is compound, consisting of a

number of interchangeable molecules or larger members, this should probably be the case, that is, a patchwork of tissues should result, as mutant and non-mutant molecules became sorted out irregularly in the cell-divisions directly following the original mutation, and an irregularly patterned mosaic adult would be formed. We may conclude that it is likely that the gene ordinarily consists of not more than one molecule, or at least not of several molecules of the same kind.

But although the adult individual arising from a mutant germ cell does not show mutant and normal tissues mixed up in any such a crazy-quilt pattern, it does, usually at any rate, contain both mutant and normal tissues, and each of these tissues usually forms, so far as can be determined, one coherent sector. The germ cells may or may not be included in the mutant sector. This fractional character of the mutants seems most plausibly interpreted on the assumption that the chromosome, or at least the individual gene, in the parental treated sperm cell was already divided into two halves, and that only one half became transmuted.[3] If this is true the sperm cell in a sense contains twice as many genes as we thought it to, and our maximum estimates of gene size must correspondingly be cut in two, bringing them even nearer to the size of a single protein molecule.

That the transmuting action of the x-rays is thus spatially narrowly circumscribed, being confined to one gene even when there are two identical genes close together, has also been shown in another way. Females were treated (fall of 1927) which contained one X-chromosome having C_l, also a normally patterned X-chromosome, and a supernumerary Y-chromosome. When this combination occurs, many offspring are formed in which, owing to non-disjunction, both X-chromosomes from the mother are present at once. It was found, on analysis of 84 such "non-disjunctional" offspring by the breeding test, that in four of them a gene in one of these treated X's had mutated, without the corresponding gene in the other treated X having been affected. Two of these mutations were lethals and two were visibles. No case of a simultaneous mutation of two corresponding genes has been found. Hence, the accidental position of the gene in the path of the rays, rather than its chemical composition *per se*, may be of major importance in determining whether it happens to become transmuted.

More direct evidence for this conclusion was found in two cases in which it could be shown that two or more near-by loci, containing genes of different character, had become changed simultaneously. The more striking of these cases is that of notched wing and mottled eye, which arose together and which lie only about one and a half units apart in the chromosome; a breakage in the same chromosome, at a point not far removed from these, also occurred at the same time. This "group muta-

tion," or simultaneous occurrence of gene changes, in this case close together in a row, suggests that the secondary or β radiation, the released electrons, may be the effective agent, and the chance position of the gene in relation to the course of the electron may be a deciding condition in the production of the mutation. That the mutations are thus caused by chance absorption of individual quanta, such as may take place in the line of these secondary rays, is further suggested by the lack of relation between the amount of the x-ray dose and the intensity or the character of the mutations produced: their number varies with the x-ray dosage, but it seems that the "degree" or nature of the individual mutations themselves does not vary with the dosage.

Intensive study of the individual mutations is however called for if we would know more of the properties of the gene. Of the individual mutations, the mottled above referred to has been the most interesting and unique, it being the only example, among our induced mutations, of an "ever-sporting gene." Weinstein has recently reported another case. The continual mutations of mottled backward and forward, even in the somatic tissue, are indicated by the splotches on the eye, and different strains of it, having more and fewer spottings, and varying intensities and extensions of color in the spottings or in the background, have been established by selection. The theory of a compound gene might at first sight seem to find a very favorable case here, but it must be recalled that other allelomorphs of this gene are known (eosin, tinge, etc.) which, although they too show a color intermediate between red and white, are not unstable at all. Thus the instability of mottled would seem to depend on some other feature of the gene structure than the apparent mixture of color elements. Moreover, the variations of mottled are to some extent correlated with variations in the notching of the wings, though the notch gene is not identical with, or even directly adjacent to the mottled gene.

There are other individual mutations which can be used for intensive study, because their frequency of occurrence after raying is so high. For example, ordinary white eye (of which mottled is an allelomorph) has been produced something like twenty times, its frequency of occurrence, after the heavy treatment, being of the order of once in every thousand germ cells. The dominant, star eye, has been found to have the same or an even higher frequency. Rudimentary wing has occurred several times, so have miniature wing and various other types.

3. *Evidence for Other Alterations than Losses.*—In view of this ready repetition of given changes, by an agent known to be so often destructive in its effect, Altenburg has suggested a question which must be faced. Are the x-rays merely punching holes in the chromosomes, or, more precisely stated, simply causing losses or inactivations of genes or gene parts? If so, the usefulness of the x-ray, both from a practical and from a theo-

retical standpoint in biology, would be seriously limited. We have therefore attempted this year (1927–1928) to discover, at the suggestion of my wife, whether mutations could be produced in both of two opposite directions by x-rays—as from red to white eye and also from white to (or toward) red. In the case of this particular gene, extensive counts by Hanson, by Patterson and by myself have so far failed to show a mutation from white to (or toward) red, though such a change would have been found had it tended to occur with as high a frequency as the red to white mutation. Hanson has found four reverse mutations of the dominant bar eye to normal round eyes, but this change would, on the basis of Sturtevant's work, be a loss, and the opposite change, round to bar, has not yet been produced. Again, in the experiment previously cited which proved that only one of two corresponding genes was transmuted at a time, females were used which were homozygous for the mutant gene scute (in which bristles are missing on the scutellum) and the two visible mutations found in this experiment both consisted of reverse mutations of this "scute" to the dominant normal gene which causes the production of scutellar bristles. This occurred once in the C_1-bearing X-chromosome and once in the other X. Unfortunately, the opposite change—normal to scute—has not yet been observed in x-rayed material, although it has occurred several times in the past in other work (as has also the scute to normal change). This case, too, then fails to quite meet the conditions for the solution of our problem.

Recently, however, unexpected evidence from another angle, bearing on the point at issue, has been obtained. In the x-ray work this year, an "eyeless" (reduced eye) mutation of appearance similar to the familiar one, and also lying in the tiny chromosomes of the fourth pair, has been discovered, but this x-ray eyeless, unlike the old eyeless, is a *dominant*, that is, it shows as "eyeless" even if one normal fourth chromosome is present in the fly. Now, previous work of Bridges has shown that if one normal chromosome of the fourth pair is present, and if the other fourth chromosome has been completely lost, the flies (called "halpo-IV's") do not show the characteristics of eyeless (being normal eyed in these respects). The new dominant eyeless, therefore, does not seem to be due to a loss, since the real loss, *per se*, is known to give no dominant eyeless effect. Nor is the new eyeless a mere "duplication," for an extra fourth chromosome, in Bridges' experiments, produced no such eyeless effect either. The dominant eyeless accordingly appears to consist of some other kind of alteration in the gene than a change purely quantitative in its nature. There is only one objection of any importance to this conclusion, though perhaps it is rather far-fetched: that is, that the loss of a part of one chromosome might perhaps produce a different and, in some respects, a stronger effect than the loss of the whole chromosome would,

on the assumption that adjacent genes (or genetic materials) somehow interact in a special way by reason of their proximity.

A stronger case, not open to any of the above objections, has even more recently been worked out (April-May, 1928).[4] Here a mutation has been found, after raying, in which the gene for forked bristles reverted to the dominant normal bristles, and true-breeding stock of the reversed gene has been established. "Contamination" can be ruled out as an explanation here, since a peculiar chromosome inversion which had been present served as a check on this. It will be recalled that mutation in the direction, normal bristles to forked, had already been observed (several times) after raying. As the forked to normal change has never been observed without raying, and as the present reverse mutation was accompanied by another visible mutation, in a different locus of the same chromosome, it is highly unlikely that it was of "spontaneous" origin (i.e., not caused by the raying). Hence this case affords convincing evidence that changes other than losses can be produced by x-rays. It will be desirable, however, to search for further such cases of induced mutations in opposite directions, in order that quantitative data may be obtained concerning the potentialities of the genes.

4. *The Reorganization of Chromosomes.*—In the earlier experiments, besides the gene mutations, numerous cases were found that seemed to be so-called "chromosome mutations," or, more properly speaking, chromosome reorganizations, as they involved rearrangements of blocks of genes comprising whole sections of chromosomes. Usually, these "mutants" showed modifications in their cross-over frequencies, such as are taken to indicate that the chromosome has been broken, and a section of it turned around in its place or otherwise reattached abnormally to the remaining section, but some of the cases clearly involved an attachment between the broken-off fragment and some other, non-homologous chromosome. In order to obtain more definite data on the latter cases, which we call "translocations"—only one example of which seems to have been definitely established in previous Drosophila work, by Bridges in 1918, though Blakeslee in 1926 reported a case found, *after raying*, in Datura—Altenburg and I have this year (1927–1928) made certain crosses of treated flies to others having dominant genes, as markers, in two of their main autosomes, and the offspring have been bred to discover such infractions of Mendel's second law, the law of "random assortment," as would result from an attachment between pieces of non-homologous chromosomes. We find that such translocations, after x-ray treatment, are nearly as abundant as are the (detectable) gene-mutations. More than 70 have so far been found, including attachments between one long autosome and another, between either long autosome and the X, and between either long autosome and the Y, and also two cases of double translocations

involving all three long chromosomes simultaneously, in such a way that the animals bred as if they had only one pair of very long chromosomes (like *Ascaris univalens*). This is the number of such double events to be expected if the different translocations really occur independently of one another. Analysis of all the data shows that there is little or no preference for attachment between particular chromosomes, it being a chance affair, except that the likelihood of a given chromosome being broken, or of its serving for the attachment of the fragment of another, varies approximately with its own length.

Breakages, and also attachments, may occur at various points along the chromosome "map," that can in many cases be determined by genetic evidence. We may, then, examine the cells under the microscope, and verify our picture as genetically drawn. This problem has now been attacked, and figures have been obtained by the writer and by Altenburg, in several cases of translocation, which seem plainly to meet the genetic expectations. However, Drosophila cytology is elusive in its finer details, and so it will be desirable to obtain a fair number of incontrovertible and mutually confirmatory figures of each case before the conformation in question can be considered as conclusively established. Professor T. S. Painter is now working intensively on a number of cases of this sort. It should be noted that such cytological verification not only establishes the claims made with respect to the individual translocations involved, but also serves to prove directly that our genetic methods of reasoning are sound: that is, that the genes really do lie in the chromosome in linear arrangement, in the physical order in which we have theoretically mapped them—a cardinal principle which not all those who believe in the chromosome theory in a more general way have hitherto admitted. And with this proof, it can be shown, goes the proof of the corollary proposition, that the so-called "mechanical theory" of crossing-over is correct.

When the individuals containing translocations are bred, various points concerning the behavior of the chromosomes come to light. It is found, for example, that the translocated piece sometimes crosses-over with its normal homologue that is differently placed; this illustrates the fact that synapsis and crossing-over depend on an attraction between like genes rather than between chromosomes as a whole. Usually, however, cross-over *frequencies* are much affected in one way or another.

Among the zygotes formed on breeding individuals containing translocations, there are always two classes of "unbalanced" types, one of which receives a chromosome deficient in a section of its genes, but fails to get the translocated section that originally belonged there, and the other of which, conversely, receives an entire normal chromosome of this category, and gets the translocated section in addition. Often the individuals of such "unbalanced" compositions die, but occasionally they live, and then

they commonly show abnormalities caused by the unusual gene proportions; thus the somatic and genetic effects of missing or extra genes may be studied. Since these abnormal or lethal individuals are regularly produced when individuals with translocated chromosomes are crossed to normals, it is evident that the existence of such translocations in homozygous condition in a sector of a population will tend to isolate it physiologically from the rest, and so might become a factor in allowing divergence in evolution. The obstacles in the way of crossing-over and of free recombination in the case of translocations (and inversions) tends in the same direction.

5. *Generality of the Effect.*—Various other workers, preëminently Weinstein, have now confirmed the effectiveness of X-rays in producing both gene mutation and chromosome reorganization in Drosophila. In our laboratory Hanson, Patterson, Oliver and Harris have obtained definitely positive results. Recently Whiting, working on the wasp Hadrobracon, has extended the principle of gene change by X-rays to another animal than Drosophila. Stadler, working on barley and maize, has conclusively demonstrated, by means of ingenious methods, both gene mutations and chromosome aberrations ("non-disjunction?") to be produced in plants by x-rays. It should be stated, in this connection, that his work was carried on simultaneously with and independently of that of the present writer. The work of Goodspeed and Olson on tobacco, in which numerous phenotypic and also chromosomal abnormalities were clearly induced in the progeny of rayed flowers,[5] now extends the principle of heritable chromosome reorganization by x-rays to plants, and there seems to be a good opportunity for the production of gene mutations by x-rays also to be proved soon in this form.

The effect of x-rays on genes seems quite general in another sense as well. That is, the production of gene mutations by this means is not confined to a particular type of cell. In Drosophila, it is not only mature germ cells of both sexes which are susceptible, but also immature cells. For, in the earlier experiments (1926), it will be recalled that mutations were found not only in individuals from eggs laid immediately after treatment, but also in those from eggs laid a week or more afterwards, which must have been in the oogonial stage when the female was treated. In one such case, moreover, two daughters of the same treated mother were found to have an identical mutation (the semi-lethal, semi-dominant "cloven thorax," located near scute); these had obviously been derived from the same treated oogonium. Professor J. T. Patterson, now working on the general problem of effects on immature cells, finds that cells of the larva and early embryo may have their genes transmuted, and chromosomes eliminated or broken, by treatment given in the early stages: thus mosaic adults are produced bearing small or large patches of visibly

mutant tissue, each patch derived from one mutant larval somatic cell, while similar phenomena in the germ cells result in groups of identically mutant offspring. Stadler has reported analogous findings of groups of identically mutant offspring from barley rayed in the seedling stage. These genetic effects on "formative tissue" may also furnish the explanation for the effects of raying on the hair color of homozygous and heterozygous mice, reported by Hance.

6. *Is the Effect Unique?*—Before closing, we may raise the question: to what extent are x-rays unique in their effects? May not other maltreatment also produce mutations? It is to be expected that radium radiation, because of the physical similarity of its effects to those of x-rays, would produce similar results in this respect also, and results obtained this spring by Hanson (using the C_l method) are clearly positive on this point. In one of his experiments the gamma rays alone were allowed to reach the flies, and in these, too, mutations were produced in abundance. Proceeding down the spectrum in the other direction, Altenburg, in the spring of 1927, tried maximum toleration doses of ultraviolet rays from a mercury arc, on an extensive scale, and the results here were definitely negative, in the sense that there could have been no effect of nearly the same order as is produced by maximum toleration doses of x-rays. Of course, individual wave-lengths still deserve trial separately.

The writer has tried five different treatments this spring (1928), in addition to controls, using approximately 500 cultures of the C_l type in each case, and examining them primarily for lethals. The controls yielded 2 lethals. One treatment consisted of the inclusion of one per cent of lead acetate in the food, on which the flies were placed from the egg stage through maturity; another treatment involved a similar protracted feeding on arsenic trioxide, in a semi-lethal concentration (0.015% of the food); in a third treatment, semi-lethal doses of manganese chloride (0.62% and 0.31% of the food) were administered in like manner. In all of these cases combined there was only one lethal (and no other) mutation observed (the lethal was in a manganese culture), and we may assume with confidence that, if the frequency of lethal mutations was low here, the general frequency of visible mutations must have been much lower still. This result, therefore, is at variance with that reported for butterflies by J. W. H. Harrison, since this author concludes that *visible* mutations are produced in abundance, both by concentrations of lead and of manganese far weaker than those here employed. Of course, it may be held that flies and butterflies would be affected totally differently.

The other two treatments tried on the flies consisted of semi-lethal doses of the vital stain, Janus green (0.25% of the food), and semi-lethal, partially sterilizing exposures to a temperature of 36°C. for from 40 to 64 hours. In the former experiments 2 lethals and 3 semi-lethals, and

in the latter experiment 4 lethal mutations were obtained. These numbers are not significantly higher, from a statistical standpoint, than the figure of 2 lethals gotten in the controls, yet there may have been a comparatively slight effect here, and a repetition of these treatments will be desirable. The apparent effect, however, is in neither case larger than that already known to be produced by moderately high temperature when this is applied for a week or more to non-radiated material (experiments of Altenburg and of the author, 1919–1926), and such an influence of temperature may conceivably only consist in rendering more effective that natural radiation which commonly penetrates even "non-radiated" cells (just as Holthusen, followed by Dognon, has found, in Ascaris eggs, that a rise in temperature may result in a marked increase in the injurious effect on development, of x-rays artificially applied). It may be repeated, however that if there was any effect in either of these two experiments, it was too small to be really demonstrable by the technique employed, and this supposititious effect must have been of an intensity at least three "octaves" lower than that of the x-ray effect.

Thus we are brought before the question: are all mutations ultimately due to rays of short wave-length and to high-speed particles of corresponding energy content? *If* so, biological evolution has been made possible only by the stray radiation present in nature—the beta and gamma rays, and the cosmic rays. This question permits a definite solution, for some organisms at any rate, if only we can compare the mutation rate in ordinary controls with that in cultures from which a large part of the natural radiation has been artificially excluded, but great numbers of cultures will be necessary for a study involving only such low mutation rates. Moreover, absolutely all radiation cannot be excluded, for, as J. B. S. Haldane has pointed out in this connection, there would always be some "residual radiation" associated with the necessary potassium of living things. In this projected experiment the coöperation of physicists will be desirable in the working out of effective means for securing the physical conditions required.

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[1] This paper includes, besides new material, most of that presented by the author at the A. A. A. S. meeting, Dec., 1927.

[2] Department of Zoölogy, Contribution No. 221.

[3] Since the above was written Dr. Robertson has shown me figures (unpublished) in which a longitudinally split condition of the chromosomes in the spermatids of grasshoppers is evident, and Dr. M. T. Harmon has informed me that she has observed the same phenomenon in the spermatids of guinea pigs. Various cytologists have observed a split condition of chromosomes in the telophase stages of other cell divisions. According to observations of Shiwago, chromosomes are double at all stages, and split secondarily just before the identical halves separate in mitosis.

[4] This paragraph has been inserted since the reading of the original paper, as the reversion was not observed until just after the author's return from the meeting. In the original paper it had been stated that no reversions of forked had been obtained. The writer wishes to thank Professor J. T. Patterson for invaluable aid in this experiment, rendered during the writer's absence on the trip.

[5] E. Stein (1922, 1926) had previously reported numerous phenotypic abnormalities of Antirrhinum, produced by radiation, which were inherited through vegetative reproduction. She also noted abnormalities of chromosome distribution occurring at the reduction division, long after treatment. In animals, non-disjunction of the X and other chromosomes at maturation, produced by radiation, was first demonstrated by Mohr (1919) in *Decticus*.

ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL

By Morris Marden[1]

Department of Mathematics, University of Wisconsin

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The purpose of this note is to indicate the nature of some results obtained by the writer in generalizing a group of theorems proved several years ago by Professor Walsh of Harvard University concerning the approximate geometric location of the roots of the derivative of a polynomial.[2] A principal result is the following

Theorem. Suppose m_i roots of a polynomial $f(z)$ of degree n have as their common locus a region C_i consisting of the interior and the circumference of the circle

$$C_i = x^2 + y^2 - 2\alpha_i x - 2\beta_i y + \gamma_i = 0,$$

where $i = 1, 2, \ldots, q+1$, *and* $\sum_{i=1}^{q+1} m_i = n$.

Then the roots of the derivative of $f(z)$ have as their locus

(*a*) *the common points of any two of the regions C_i;*

(*b*) *the points of every region C_i for which $m_i \neq 1$; and*

(*c*) *the interior and the boundary of all the ovals of the q-circular $2q$-ic curve*

$$\sum_{i=1}^{q+1} \frac{nm_i}{C_i} - \sum_{\substack{i=1 \\ j=i+1}}^{q+1} \frac{m_i m_j \sigma_{ij}}{C_i C_j} = 0, \qquad (1)$$

where σ_{ij} is the square of the common external tangent of the circles C_i and C_j.[3]

The singular foci of (1) *lie at the roots of the logarithmic derivative of*

$$g(z) = \prod_{j=1}^{q+1} (z - \delta_j)^{m_j},$$

where $\delta_j = \alpha_j + i\beta_j$.

If no two of the regions (*c*), $C_1, C_2, \ldots, C_{q+1}$ *have any point in common, they contain respectively* $q, m_1 - 1, m_2 - 1, \ldots, m_{q+1} - 1$ *roots of* $f'(z)$. *If, in addition,* (*c*) *falls into* q *distinct parts, each part contains just one root of* $f'(z)$.

This theorem may be proved by a method of "iterated envelopes."[4] First, only the m_1 roots of $f(z)$ within C_1 are permitted to vary, all the remaining roots of $f(z)$ being held fast, and the locus R_1 of the roots of $f'(z)$ under these conditions is found. Then the envelope R_2 of the R_1 when, in addition, the m_2 roots of $f(z)$ within C_2 are allowed to vary is obtained; and so on. Finally, by mathematical induction, the envelope R_{N+1} of the R_N when the m_{N+1} roots of $f(z)$ within C_{N+1} are also made to vary is secured. The proof of the theorem is completed by setting $N = q$.

The theorem has been verified in Professor Walsh's special cases, and a similar theorem obtained for the case that some or all of the regions C_i are the exteriors of circles. By essentially the same methods, some of Professor Walsh's results in connection with the location of the roots of the Jacobian of two binary forms and of the derivative of a rational function[5] have also been generalized analytically to any number of circular regions C_i.

The writer hopes presently to be able to report upon similar problems involving higher derivatives of a polynomial and linear combinations of these derivatives.

[1] NATIONAL RESEARCH FELLOW.

[2] Walsh, J. L., *Comptes Rendus du Congrès International des Mathématiciens*, Strasbourg, 1920; these PROCEEDINGS, **8** (1922), 139–141; *Trans. Amer. Math. Soc.*, **24** (1922), 52.

[3] In case the two circles C_i and C_j do not have a common external tangent, we shall use the expression

$$\sigma_{ij} = d_{ij}^2 - (r_i - r_j)^2,$$

where r_i and r_j are the radii of C_i and C_j, respectively, and d_{ij} is the distance between the centers of C_i and C_j.

[4] Cf. Walsh, J. L., these PROCEEDINGS, **8** (1922), p. 140, and *Trans. Amer. Math. Soc.*, **22** (1921), p. 102, footnote.

[5] Walsh, J. L., *Trans. Amer. Math. Soc.*, **19** (1918), 291–8; **22** (1921), 101–16; and **24** (1922), 31–69.

CONCERNING THE *$\mathfrak{G}$ GROUP OF TRANSFORMATIONS

BY TRACY YERKES THOMAS

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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1. The projective theory of the affinely connected n-dimensional space of affine connection $\Gamma^i_{\alpha\beta}$ is equivalent to the affine theory of an $n+1$-dimensional space of affine connection ${}^*\Gamma^i_{\alpha\beta}$ under a derived group of transformations of coördinates. The connection ${}^*\Gamma^i_{\alpha\beta}$ is defined in an invariantive manner in accordance with the equations

$$
\begin{aligned}
{}^*\Gamma^i_{\alpha\beta} &= {}^*\Gamma^i_{\beta\alpha} = \Pi^i_{\alpha\beta} && (i, \alpha, \beta = 1, 2, \ldots, n)\\
{}^*\Gamma^i_{\alpha 0} &= {}^*\Gamma^i_{0\alpha} = -\frac{\delta^i_\alpha}{n+1} && (i, \alpha, = 0, 1, 2, \ldots, n)\\
{}^*\Gamma^0_{\alpha\beta} &= {}^*\Gamma^0_{\beta\alpha} = \left(\frac{n+1}{n-1}\right)\mathfrak{B}^\sigma_{\alpha\beta\sigma} && (\alpha, \beta = 1, 2, \ldots, n)
\end{aligned}
$$

where $\Pi^i_{\alpha\beta}$ is the projective connection and $\mathfrak{B}^i_{\alpha\beta\gamma}$ is the equi-projective curvature tensor. The transformation group *$\mathfrak{G}$ is given by

$$x^0 = \bar{x}^0 + \log \Delta, \quad x^i = f^i(\bar{x}^1, \ldots, \bar{x}^n)$$

in which the f^i with $i = 1, \ldots, n$ are arbitrary analytic functions of their arguments and Δ denotes the jacobian determinant

$$\begin{vmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} & \cdots & \dfrac{\partial x^1}{\partial \bar{x}^n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x^n}{\partial \bar{x}^1} & \cdots & \dfrac{\partial x^n}{\partial \bar{x}^n} \end{vmatrix}.$$

It is assumed that Δ does not vanish identically.

The functions ${}^*\Gamma^i_{\alpha\beta}$ transform by the equations

$${}^*\bar{\Gamma}^\sigma_{\alpha\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma} = \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} + {}^*\Gamma^i_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta} \qquad (1.1)$$

under a transformation of coördinates belonging to the group *$\mathfrak{G}$. The equations (1.1) have the same form as the equations of transformation of the ordinary affine connection $\Gamma^i_{\alpha\beta}$ of the n-dimensional space.[1]

2. Suppose that ${}^*\Gamma^i_{\alpha\beta}$ and ${}^*\bar{\Gamma}^i_{\alpha\beta}$ are two connections which are transformable by (1.1) by some transformation of coördinates

$$x^i = \varphi^i(\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^n), \qquad (i = 0, 1, \ldots, n). \tag{2.1}$$

The question then arises if the transformation (2.1) belongs necessarily to the group ${}^*\mathfrak{G}$. If such is the case then the problem of the projective equivalence of two affinely connected spaces has been reduced *in every detail* to the problem of the affine equivalence of two affinely connected spaces. To consider this question we derive several particular sets of equations from (1.1). If we put $\beta = 0$ in (1.1) we obtain

$$\frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^0} = -\frac{1}{n+1}\frac{\partial \bar{x}^i}{\partial \bar{x}^\alpha} - {}^*\Gamma^i_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\nu}{\partial \bar{x}^0}. \tag{2.2}$$

Next put $i = \beta$ in (1.1) and sum with $\alpha \neq 0$. This gives

$$\frac{\partial x^0}{\partial \bar{x}^\alpha} = \frac{\partial \log (x\,\bar{x})}{\partial \bar{x}^\alpha} \qquad (\alpha \neq 0)$$

where

$$(x\bar{x}) = \begin{vmatrix} \frac{\partial x^0}{\partial \bar{x}^0} & \frac{\partial x^0}{\partial \bar{x}^1} & \cdots & \frac{\partial x^0}{\partial \bar{x}^n} \\ \frac{\partial x^1}{\partial \bar{x}^0} & \frac{\partial x^1}{\partial \bar{x}^1} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x^n}{\partial \bar{x}^0} & \frac{\partial x^n}{\partial \bar{x}^1} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{vmatrix}.$$

If we had taken $\alpha = 0$ we would have obtained

$$\frac{\partial x^0}{\partial \bar{x}^0} = 1 + \frac{\partial \log (x\bar{x})}{\partial \bar{x}^0}.$$

The above equations can, therefore, be combined into the single set of equations

$$\frac{\partial x^0}{\partial \bar{x}^\alpha} = \delta^0_\alpha + \frac{\partial \log (x\bar{x})}{\partial \bar{x}^\alpha} \tag{2.3}$$

in which there is no restriction on the index α.

Let us impose the condition that

$$\frac{\partial x^i}{\partial \bar{x}^0} = \delta^i_0, \text{ (for } x^i = q^i) \tag{2.4}$$

which is a necessary condition for (2.1) to belong to the group $^*\mathfrak{G}$. Then (2.2) gives

$$\frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^o} = 0, \text{ (for } x^i = q^i).$$

Differentiating (2.2) and using (2.4) we have

$$\frac{\partial^3 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta \partial \bar{x}^o} = 0, \text{ (for } x^i = q^i).$$

Continuing, we see that all derivatives of $\partial x^i/\partial \bar{x}^o$ vanish at $x^i = q^i$ so that we have

$$\frac{\partial x^i}{\partial \bar{x}^o} = \delta^i_o$$

throughout the neighborhood of $x^i = q^i$. The determinant $(x\bar{x})$, therefore, becomes the above determinant Δ and by integration of (2.3) we obtain

$$x^o = \bar{x}^o + \log \Delta + \text{const.}$$

The constant in this equation can be taken to have the value zero since it is of no significance in the transformation of the connections $^*\Gamma^i_{\alpha\beta}$ and $^*\bar{\Gamma}^i_{\alpha\beta}$ by (1.1). This proves the following[1]

Theorem. The transformation (2.1) *satisfying* (1.1) *belongs to the group* $^*\mathfrak{G}$ *if the condition* (2.4) *is satisfied.*

We next investigate the extent to which the condition (2.4) can be removed.

3. Consider the sequence of sets of equations

$$\begin{aligned} {}^*\bar{A}^\sigma_{\alpha\beta\gamma}\, u^i_\sigma &= {}^*A^i_{\mu\nu\rho}\, u^\mu_\alpha u^\nu_\beta u^\rho_\gamma \\ {}^*\bar{A}^\sigma_{\alpha\beta\gamma\delta}\, u^i_\sigma &= {}^*A^i_{\mu\nu\rho\sigma}\, u^\mu_\alpha u^\nu_\beta u^\rho_\gamma u^\sigma_\delta \end{aligned} \qquad (3.1)$$

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which give the transformation of the projective normal tensors $^*A$ where we have put

$$u^i_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha}.$$

Suppose that the equations (3.1) have a numerical solution

$$x^i = q^i; \quad \bar{x}^i = \bar{q}^i; \quad u^i_\alpha = a^i_\alpha.$$

By making use of an argument involving normal coördinates[1] there is then determined a transformation (2.1) satisfying (1.1) which associates the point $x^i = q^i$ with the point $\bar{x}^i = \bar{q}^i$ and which is such that $\partial x^i/\partial \bar{x}^\alpha = a^i_\alpha$ for $x^i = q^i$. This shows that all conditions imposed by (1.1) on the quantities u^i_α are given by the sequence (3.1).

It can be shown that the sequence

$$\begin{aligned} {}^*\overline{B}^\sigma_{\alpha\beta\gamma}\, u^i_\sigma &= {}^*B^i_{\mu\nu\rho}\, u^\mu_\alpha u^\nu_\beta u^\rho_\gamma \\ {}^*\overline{B}^\sigma_{\alpha\beta\gamma\delta}\, u^i_\sigma &= B^i_{\mu\nu\rho\sigma}\, u^\mu_\alpha u^\nu_\beta\, u^\rho_\gamma\, u^\sigma_\delta \end{aligned} \qquad (3.2)$$

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expressing the transformation equations of the projective curvature tensor ${}^*B^i_{\alpha\beta\gamma}$ and its successive covariant derivatives is completely equivalent to the sequence (3.1).[2] In terms of this latter sequence of sets of equations we can therefore state the following

Lemma. The equations (3.2) *give all the conditions on the quantities* u^i_α *imposed by* (1.1).

The reason for using the sequence (3.2) rather than (3.1) in the statement of the above lemma is because any tensor ${}^*B$ can be obtained by a very simple process, namely, the process of covariant differentiation, from the tensor which immediately proceeds it in the sequence (3.2) whereas this is not the case for the projective normal tensors ${}^*A$.

The conditions put on the quantities u^i_α by (1.1) by which we may hope to remove to some extent the above conditions (2.4) must therefore be given by the sequence (3.2). Of the equations of the sequence (3.2) a finite number will of course give all the conditions on the quantities u^i_α.

4. Let us first take the special case where the n-dimensional space of affine connection $\Gamma^i_{\alpha\beta}$ is projective-plane. Then the projective curvature tensor ${}^*B^i_{\alpha\beta\gamma}$ vanishes identically and the space of affine connection $\overline{\Gamma}^i_{\alpha\beta}$ must likewise be projective-plane, i.e., the tensor ${}^*\overline{B}_{\alpha\beta\gamma}$ must also vanish identically if the two spaces are to be transformable by (1.1). Assuming this to be the case all equations of the sequence (3.2) reduce to $0 = 0$, and hence are satisfied by arbitrary quantities x^i_α. We may therefore select the u^i_α so that $u^i_o = \delta^i_o$ for $x^i = q^i$ and hence we have the

Theorem. The connections ${}^*\Gamma^i_{\alpha\beta}$ *and* ${}^*\overline{\Gamma}^i_{\alpha\beta}$ *are transformable by* (1.1) *by a transformation of coördinates belonging to the group* ${}^*\mathfrak{G}$ *if the spaces with connections* $\Gamma^i_{\alpha\beta}$ *and* $\overline{\Gamma}^i_{\alpha\beta}$, *respectively, are projective-plane.*

We can however select the u^i_α so that $u^i_o \neq \delta^i_o$ for $x^i = q^i$ with the result that the above connections ${}^*\Gamma^i_{\alpha\beta}$ and ${}^*\overline{\Gamma}^i_{\alpha\beta}$ for the projective-plane case are transformable by a transformation of coördinates (2.1) which does not belong to the group ${}^*\mathfrak{G}$.

5. In the following we shall consider that the space with affine connection $\Gamma^i_{\alpha\beta}$ is not projective-plane so that all the components of the projective curvature tensor ${}^*B^i_{\alpha\beta\gamma}$ do not vanish. All components of the tensor ${}^*\overline{B}^i_{\alpha\beta\gamma}$ can, therefore, not vanish if the two spaces are to be transformable.

Making use of the equations

$${}^*B^i_{\alpha\beta\gamma} = \frac{\partial {}^*\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial {}^*\Gamma^i_{\alpha\gamma}}{\partial x^\beta} + {}^*\Gamma^i_{\sigma\gamma}\,{}^*\Gamma^\sigma_{\alpha\beta} - {}^*\Gamma^i_{\sigma\beta}\,{}^*\Gamma^\sigma_{\alpha\gamma}$$

it follows immediately that

$${}^*B^i_{\alpha\beta\gamma} = 0, \qquad (\text{if } \alpha, \beta \text{ or } \gamma = 0). \tag{5.1}$$

By putting α, β and γ in turn equal 0 in the first set of equations (3.2) we, therefore, have

$$\sum_{k=1}^{n} {}^*B^i_{k\beta\gamma}\,u^k_o = 0; \quad \sum_{k=1}^{n} {}^*B^i_{\alpha k\gamma}\,u^k_o = 0; \quad \sum_{k=1}^{n} {}^*B^i_{\alpha\beta k}\,u^k_o = 0. \tag{5.2}$$

Analogous equations hold for the tensor ${}^*B^i_{\alpha\beta\gamma\delta}$. Let us observe that

$${}^*B^i_{o\beta\gamma\delta} = \frac{{}^*B^i_{\delta\beta\gamma}}{n+1}; \quad {}^*B^i_{\alpha o\gamma\delta} = \frac{{}^*B^i_{\alpha\delta\gamma}}{n+1}; \tag{5.3}$$

$${}^*B^i_{\alpha\beta o\delta} = \frac{{}^*B^i_{\alpha\beta\delta}}{n+1}; \quad {}^*B^i_{\alpha\beta\gamma o} = \frac{-2\,{}^*B^i_{\alpha\beta\gamma}}{u+1}$$

which follow immediately from the equations defining the covariant derivative of the tensor ${}^*B^i_{\alpha\beta\gamma}$. Now put $\alpha = 0$ in the second set of equations (3.2). This gives

$$\frac{{}^*\overline{B}^\sigma_{\delta\beta\gamma}}{n+1}\,u^i_\sigma = {}^*B^i_{\mu\rho\sigma}\,u^\mu_o u^\rho_\beta u^\sigma_\gamma u^\sigma_\delta. \tag{5.4}$$

Putting $\beta = 0$ we have a set of equations which can be written

$$\frac{{}^*\overline{B}^{\sigma}_{\delta\beta\gamma}}{n+1}\, u^i_\sigma = {}^*B^i_{\sigma\mu\rho\nu} u^\mu_o\, u^\nu_\beta u^\rho_\gamma u^\sigma_\delta. \tag{5.5}$$

Subtracting (5.4) and (5.5) we obtain

$$({}^*B^i_{\mu\nu\rho\sigma} + {}^*B^i_{\sigma\rho\mu\nu})\, u^\mu_o = 0$$

where use has been made of the fact that ${}^*B^i_{\mu\nu\rho}$ and, therefore, ${}^*B^i_{\mu\nu\rho\sigma}$ are skew-symmetric in the indices ν,ρ. The equations become

$$\left(\sum_{k=1}^{n} {}^*B^i_{k\nu\rho\sigma} + \sum_{k=1}^{n} {}^*B^i_{\sigma\rho k\nu}\right) u^k_o = 0 \tag{5.6}$$

since the coefficient of u^o_o vanishes by the skew-symmetric property. In the same way if we put $\gamma = 0$ in the second set of equations (3.2) and subtract from (5.4) we find

$$\left(\sum_{k=1}^{n} {}^*B^i_{k\nu\rho\sigma} + \sum_{k=1}^{n} {}^*B^i_{\sigma k\nu\rho}\right) u^k_o = 0. \tag{5.7}$$

Finally, putting $\delta = 0$ in the second set of equations (3.2), multiplying (5.4) by 2 and then adding the two sets of equations leads to equations which can be written

$$\left(2\sum_{k=1}^{n} {}^*B^i_{k\nu\rho\sigma} + \sum_{k=1}^{n} {}^*B^i_{\sigma\nu\rho k}\right) u^k_o = 0. \tag{5.8}$$

If we add (5.6), (5.7) and (5.8), make use of the identity

$${}^*B^i_{\sigma k\nu\rho} + {}^*B^i_{\sigma\nu\rho k} + {}^*B^i_{\sigma\rho k\nu} = 0 \tag{5.9}$$

and then take account of the equations (5.6), (5.7), (5.8) individually we obtain

$$\sum_{k=1}^{n} {}^*B^i_{k\nu\rho\sigma}\, u^k_o = \sum_{k=1}^{n} {}^*B^i_{\mu k\rho\sigma}\, u^k_o = \sum_{k=1}^{n} {}^*B^i_{\mu\nu k\sigma}\, u^k_o = \sum_{k=1}^{n} {}^*B^i_{\mu\nu\rho k}\, u^k_o = 0. \tag{5.10}$$

These are the equations analogous to (5.2) which we set out to deduce.

6. If we put $\alpha = 0$ in the second set of equations (3.2) we have

$${}^*\overline{B}^{\sigma}_{\alpha\beta\gamma\delta}\, u^i_\sigma = {}^*B^i_{\sigma\nu\rho\sigma}\, u^\sigma_o\, u^\nu_\beta u^\rho_\gamma u^\sigma_\delta$$

by (5.10). Or we can write

$$(1 - u^o_o)\, {}^*\overline{B}^i_{\alpha\beta\gamma} = 0$$

by (5.3) and the first set of equations (3.2). Since we are assuming that the space is not projective-plane all components ${}^*\overline{B}^i_{\alpha\beta\gamma}$ cannot vanish with the result that

$$u^o_o = 1. \tag{6.1}$$

This reduces the conditions (2.4) to a certain extent.

7. Let us now consider the particular case where the dimensional number $n = 2$. The first set of equations (3.2) can be written

$$\sum_{k=1}^{n} {}^*\overline{B}^i_{\alpha\beta\gamma}\, u^i_k + {}^*\overline{B}^o_{\alpha\beta\gamma}\, u^i_o = {}^*B^i_{\mu\nu\rho}\, u^\mu_\alpha u^\nu_\beta u^\rho_\gamma. \tag{7.1}$$

It is a known fact that for $n = 2$ all components ${}^*B^i_{\mu\nu\rho}$ vanish identically if the indices i, μ, ν, ρ have values 1, 2. This fact combined with (5.1) shows that the equations (7.1) for $n = 2$ give

$${}^*\overline{B}^o_{\alpha\beta\gamma}\, u^i_o = 0, \qquad (i, \alpha, \beta, \gamma = 1, 2). \tag{7.2}$$

But the vanishing of the components ${}^*\overline{B}^o_{\alpha\beta\gamma}$ in (7.2) is the condition for the space to be projective-plane. Since we are assuming that this is not the case we have from (7.2) that

$$u^i_o = 0, \quad (i = 1, 2).$$

By (6.1) the quantity $u^o_o = 1$ so that the condition (2.4) is automatically satisfied for $n = 2$. This gives the following

Theorem. If $n = 2$ and the spaces are not projective-plane any transformation (2.1) *satisfying* (1.1) *belongs to the group* *$\mathfrak{G}$.

8. Going to the general case $n \geqslant 3$ we can regard the equations (5.2) and (5.10) as imposing conditions on the quantities u^k_o $(k = 1, \ldots, n)$. If we can find n equations from the sets (5.2) and (5.10) for which the determinant D formed from the coefficients of the u^k_o does not vanish we must have $u^k_o = 0$. Since this non-vanishing determinant D will exist in general we have the following

Theorem. If $n \geqslant 3$ and the spaces are not projective-plane a transformation (2.1) *satisfying* (1.1) *will in general belong to the group* *$\mathfrak{G}$. *More particularly the transformation* (2.1) *will belong to the group* *$\mathfrak{G}$ *if the above non-vanishing determinant D exists.*

It is evident that equations analogous to (5.2) and (5.10) which will also give conditions on the quantities u^k_o $(k = 1, \ldots, n)$ can be deduced. However, it is likely that the quantities u^k_o will always vanish for $n \geqslant 3$ as we have proved to be the case for $n = 2$. An investigation of this question would be desirable.

[1] *Math. Zeitschrift*, Vol. **25** (1926), pp. 723–733.

[2] See *Annals of Math.*, Vol. **28** (1927), p. 657, where the analogous theorem is proved for the quadratic differential form.

CONFORMAL TENSORS AND CONNECTIONS

BY OSWALD VEBLEN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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In this note I attempt to set forth the system of invariants which is appropriate to the conformal geometry of Riemannian spaces. It is a development in the direction of my recent paper on projective tensors and connections[1] of the ideas established by T. Y. Thomas[2,3] and J. M. Thomas[4,5] in four papers in these PROCEEDINGS. A class of invariants is found which I call *conformal tensors* which have $(n+2)^k$ components in each coördinate system and a linear law of transformation with coefficients determined by a certain group which I call the *enlarged conformal group*. This group has to the conformal group much the same relation that the affine group has to the Euclidean group.

It does not seem to be possible to find an invariant with $(n+2)^3$ components which will play the rôle of an affine connection in covariant differentiation. But by using one with $(n+2)^2(n+1)$ components analogous to that already used by T. Y. Thomas[3] and to the set of $(n+2)^2 n$ components used by J. A. Schouten[6] it is possible to find most of the components of a "conformal derivative" of any conformal tensor and then to determine the remaining components by imposing an invariant condition on the tensor sought. We thus have a recursion process which generates an infinite sequence of conformal tensors from any given conformal tensor. Thus to get a complete sequence of invariants for a conformal geometry it will be sufficient to start with the conformal curvature tensor (whose components include those of Weyl's[7] conformal curvature tensor) and apply this process of conformal differentiation. These invariants would seem to be quite suitable for the complete development of the theory of conformal geometry outlined from a somewhat different point of view by E. Cartan.[8]

1. *The Fundamental Relative Tensor.*—According to an observation by T. Y. Thomas a conformal geometry of n dimensions is really the theory of a relative tensor of weight $-2/n$. We start with a family of Riemannian metrics all of which determine the same angle at each point. The formula for the differential of arc of these metrics is

$$ds^2 = \sigma g_{ij}\, dx^i\, dx^j \qquad (1.1)$$

in which σ is an arbitrary scalar and g_{ij} an absolute covariant tensor of the second order. In a given coördinate system, there is a unique one of these metrics, namely,

$$ds^2 = \frac{g_{ij}}{g^{\frac{1}{n}}} dx^i dx^j = G_{ij} dx^i dx^j \tag{1.2}$$

for which the determinant of the coefficients is unity. The invariant which has these coefficients as its components in each coördinate system is a relative tensor of weight $-2/n$, that is to say it has the law of transformation

$$\bar{G}_{ij} = u^{-\frac{2}{n}} G_{ab} u_i^a u_j^b \tag{1.3}$$

in which

$$u_j^i = \frac{\partial x^i}{\partial \bar{x}^j} \text{ and } u = \left|\frac{\partial x}{\partial \bar{x}}\right|. \tag{1.4}$$

With it there is associated the relative contravariant tensor of weight $2/n$,

$$G^{ij} = g^{\frac{1}{n}} g^{ij}$$

such that

$$G^{ij} G_{ik} = \delta_k^j.$$

The determinant $|G^{ij}|$ is, of course, equal to unity.

2. *The Conformal Connection* is an invariant introduced by J. M. Thomas[4] which has the law of transformation,

$$K_{jk}^i = K_{bc}^a v_a^i u_j^b u_k^c + \frac{\partial u_j^a}{\partial x^k} v_a^i - \frac{1}{n} (\delta_j^i u_k^o + \delta_k^i u_j^o - G_{jk} G^{ia} u_a^o) \tag{2.1}$$

in which

$$u_j^o = \frac{\partial \log u}{\partial \bar{x}^j} \text{ and } v_j^i = \frac{\partial \bar{x}^i}{\partial x^j}.$$

It is connected with the tensor G_{ij}, as was remarked by T. Y. Thomas, by the same formula as defines the Chrystoffel symbols of the second kind

$$K_{jk}^i = \frac{1}{2} G^{ia} \left(\frac{\partial G_{aj}}{\partial x^k} + \frac{\partial G_{ak}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^a} \right). \tag{2.2}$$

In the Euclidean case its components vanish in all coördinate systems for which the components of the tensor G_{ij} are constants. The group of all transformations between these coördinate systems is the *enlarged conformal group*. It contains the conformal groups, each of which leaves a set of constant G's unaltered, as a family of mutually conjugate subgroups. Its relations with the conformal group are thus analogous to those of the affine group with the Euclidean group.

The theory of a conformal connection, i.e., of an invariant with the law of transformation (2.2) is a distinct geometry which may be called the *enlarged conformal geometry*.

The equations of the enlarged conformal group are the solutions of the differential equations obtained by setting the quantities K and $\overline{K}$ in (2.1) equal to zero. This is analogous to what happens in the affine and projective geometries; if we apply the same process to the affine and projective connections, respectively, we find the affine and the projective groups, respectively. The only difference is that the equations of the enlarged conformal group are more complicated. Before deriving them, let us find the equations for the conformal groups.

3. *The Conformal Group.*—A Euclidean conformal group is the group of all transformations which do not alter the components G_{ij} of a relative tensor of weight $-2/n$, provided these components are constants.

Such transformations can be set up by introducing coördinates analogous to tetracyclic coördinates. Let

$$x^{n+1} = \frac{1}{2} G_{ij} x^i x^j \tag{3.1}$$

and let $z^0, z^1, \ldots, z^{n+1}$ be homogeneous coördinates such that

$$\frac{z^A}{z^o} = x^A \qquad (A = 1, 2, \ldots, n+1).$$

The relation (3.1) may then be written in the form

$$G_{\alpha\beta} z^\alpha z^\beta = 0 \qquad (\alpha,\beta = 0,1, \ldots, n+1) \tag{3.2}$$

provided we agree that

$$\begin{gathered} G_{o(n+1)} = G_{(n+1)o} = -1, \\ G_{io} = G_{oi} = G_{(n+1)i} = G_{i(n+1)} = G_{oo} = G_{n+1,\,n+1} = 0 \qquad (i = 1, 2, \ldots, n). \end{gathered} \tag{3.3}$$

The matrix of the quantities $G_{\alpha\beta}$ is that given by T. Y. Thomas on page 358, Vol. 12 of these PROCEEDINGS.

Since we have to use indices running over three ranges, we adopt the conventions that: small Roman letters are used for indices on the range $1, 2, \ldots, n$; capital Roman letters on the range $1, 2, \ldots, n, n+1$; and Greek letters on the range $0, 1, 2, \ldots, n+1$.

The equation (3.2) is undisturbed by linear transformations

$$z^\alpha = f^\alpha_\beta z^\beta \tag{3.4}$$

provided

$$hG_{\alpha\beta} = G_{\sigma\tau} f^\sigma_\alpha f^\tau_\beta = G_{ij} f^i_\alpha f^j_\beta - f^o_\alpha f^{n+1}_\beta - f^o_\beta f^{n+1}_\alpha \tag{3.5}$$

where h is any constant. But on forming the determinants of the quantities on the two sides of (3.5) we see that

$$h = |f_\beta^\alpha|^{\frac{2}{n+2}}$$

where $|f_\beta^\alpha|$ is the determinant of the $(n+2)^2$ quantities f_β^α. The group defined by (3.4) and (3.5) is the Euclidean conformal group which has been studied by Darboux and others by the use of more special coördinates. It has a sub-group consisting of those transformations which leave the origin (i.e., the point $z^A = 0$) invariant. For these transformations,

$$f_o^A = 0,$$

and we may assume without loss of generality that

$$f_o^o = 1.$$

The conditions (3.5) then reduce to

$$f_j^{n+1} = 0 \qquad \text{and} \qquad h = f^{\frac{2}{n}}$$

where f is the n-rowed determinant

$$f = |f_j^i|, \tag{3.6}$$

and also to

$$G_{ij} = f^{-\frac{2}{n}} G_{pq} f_i^p f_j^q, \tag{3.7}$$

$$f_{n+1}^{n+1} = f^{\frac{2}{n}}, f^o = f^{-\frac{2}{n}} G_{pq} f_i^p f_{n+1}^q, 2f_{n+1}^o = f^{-\frac{2}{n}} G_{pq} f_{n+1}^p f_{n+1}^q. \tag{3.8}$$

4. *The Enlarged Conformal Group* is the group of all transformations between coördinate systems in which the components G_{ij} are constants. For our purposes we need consider only the sub-group G of transformations between coördinate systems of this type with the same origin. Given any such coördinate system, the components G_{ij} in this coördinate system are determined and thus the $n+2$ homogeneous coördinates z are determined. If we also specify $n^2 + n$ constants f_A^i, such that $f \neq 0$ (cf. (3.6)), we obtain a matrix

$$\begin{matrix}
1 & f_1^0 & f_2^0 & \cdots & f_n^0 & f_{n+1}^0 \\
0 & f_1^1 & f_2^1 & \cdots & f_n^1 & f_{n+1}^1 \\
0 & f_1^2 & f_2^2 & \cdots & f_n^2 & f_{n+1}^2 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & f_1^n & f_2^n & \cdots & f_n^n & f_{n+1}^n \\
0 & 0 & 0 & \cdots & 0 & f^{\frac{2}{n}}
\end{matrix} \tag{4.1}$$

in which the elements of the first row are determined by the equations (3.8). The elements of this matrix are the coefficients of a transformation

$$z^\alpha = f^\alpha_\beta \bar{z}^\beta. \tag{4.2}$$

This transformation obviously carries the components $G_{\alpha\beta}$ into another set of constant components which satisfy the conditions (3.3) and is therefore a transformation of our group. The components of the relative tensor in the coördinate system $\bar{x}$ so defined are

$$\bar{G}_{ij} = f^{-\frac{2}{m}} G_{ab} f^a_i f^b_j. \tag{4.3}$$

It is important to observe that the components G_{ij} in the coördinate system x are implicit in (4.1). Thus there is a family of matrices (4.1) determined by each coördinate system in which the components of the relative tensor are constants.

The equations (3.8) can be converted by (4.3) into

$$f^j_{n+1} = \bar{G}^{pq} f^j_p f^o_q, \quad 2 f^o_{n+1} = \bar{G}^{pq} f^o_p f^o_q, \tag{4.4}$$

which shows that the elements of the second to the $(n+1)$st columns of (3.1) can be chosen arbitrarily and the elements of the last column then determined.

If we combine (4.2) with a transformation

$$\bar{z}^\alpha = \tilde{f}^\alpha_\beta \tilde{z}^\beta$$

determined by the numbers $\bar{G}_{ij}$ and an arbitrary set of $n^2 + n$ numbers $\tilde{f}^i_A$ such that $\tilde{f} \neq 0$, we find

$$z^\alpha = \tilde{f}^\alpha_\beta \tilde{z}^\beta$$

in which

$$\tilde{f}^\alpha_\beta = f^\alpha_\gamma \tilde{f}^\gamma_\beta.$$

Moreover, $\tilde{f}^\alpha_\beta$ are the elements of a matrix analogous to (4.1) built out of the quantities G_{ij} and $\tilde{f}^i_A$ such that

$$f^i_A = f^i_k \tilde{f}^k_A.$$

Thus any coördinate system z, obtained by first transforming z to $\tilde{z}$ by a transformation with matrix of the type (4.1) and then transforming from $\tilde{z}$ to $\tilde{z}$ with an analogous matrix, may be obtained directly by a transformation with matrix of type (4.1) from z to $\tilde{z}$. It is easily verified also that if $\tilde{z}$ and $\tilde{z}$ are two coördinate systems obtained from z by transformations with matrices of type (4.1) the transformation from $\tilde{z}$ to $\tilde{z}$ is of analogous type. Hence, all the coördinate systems obtainable from the z coördinate system by transformations (4.2) are related among themselves by transformations of analogous type. The group of all these transformations is obviously a sub-group of our group G. It is in

fact identical with G, as can be verified by carrying out the solution of the differential equations obtained by setting K and $\bar{K}$ equal to zero in (2.1).

In terms of the variables $x^1, x^2, \ldots, x^n, x^{n+1}$, the transformations of our group take the form

$$x^A = \frac{f^A_B \bar{x}^B}{1 + f^o_A \bar{x}^A}. \tag{4.5}$$

In terms of the original variables $x^1, x^2, \ldots, x^n$, we have the transformations,

$$x^i = \frac{f^i_j \bar{x}^j + \frac{1}{2} f^i_{n+1} \bar{G}_{pq} \bar{x}^p \bar{x}^q}{1 + f^o_j \bar{x}^j + \frac{1}{2} f^o_{n+1} \bar{G}_{pq} \bar{x}^p \bar{x}^q}. \tag{4.6}$$

The one of the equations (4.5) for which $A = n + 1$ is

$$G_{ij} x^i x^j = \frac{f^n \bar{G}_{pq} \bar{x}^p \bar{x}^q}{1 + f^o_j \bar{x}^j + \frac{1}{2} f^o_{n+1} \bar{G}_{pq} \bar{x}^p \bar{x}^q} \tag{4.7}$$

which is a consequence of (4.6), (4.3) and (4.4).

Let us introduce the notation

$$x_i = G_{ij} x^j = \frac{\partial x^{n+1}}{\partial x^i}. \tag{4.8}$$

Then on differentiating (4.6) we find

$$\frac{\partial x^i}{\partial \bar{x}^j} = \frac{f^i_j + f^i_{n+1} \bar{x}_j - x^i (f^o_j + f^o_{n+1} \bar{x}_j)}{1 + f^o_A \bar{x}^A} \tag{4.9}$$

and see that the functional determinant of (4.6) is

$$\left| \frac{\partial x}{\partial \bar{x}} \right| = \frac{1}{(1 + f^o_A \bar{x}^A)^n} \begin{vmatrix} 1 & f^o_1 & f^o_2 \cdots f^o_n & f^o_{n+1} \\ x^1 & f^1_1 & f^1_2 \cdots f^1_n & f^1_{n+1} \\ x^2 & f^2_1 & f^2_2 \cdots f^2_n & f^2_{n+1} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ x^n & f^n_1 & f^n_2 \cdots n f^n_n & f^n_{n+1} \\ 0 & -\bar{x}_1 & -x_2 \cdots -\bar{x}_n & 1 \end{vmatrix} = \frac{f}{(1 + f^o_A \bar{x}^A)^n}. \tag{4.10}$$

It then follows by a rather direct calculation that for the transformations (4.6),

$$\left|\frac{\partial x}{\partial \bar{x}}\right|^{-\frac{2}{n}} G_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} = f^{-\frac{2}{n}} G_{ij} f^i_p f^j_q = \bar{G}_{pq}. \tag{4.11}$$

We also find

$$\frac{\partial \log \left|\frac{\partial x}{\partial \bar{x}}\right|}{\partial \bar{x}^j} = -n \frac{f^o_j + f^o_{n+1} \bar{x}_j}{1 + f^o_A \bar{x}^A}. \tag{4.12}$$

5. There is an obvious isomorphism between the sub-group of the enlarged conformal group which leaves one point invariant and the group of all analytic transformations. Let

$$x^i = u^i(\bar{x}) \tag{5.1}$$

be an arbitrary analytic transformation and let

$$f^i_j = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_o, \qquad u = \left|\frac{\partial x}{\partial \bar{x}}\right|$$

$$f^\alpha_o = \delta^\alpha_o, \qquad f^o_j = -\frac{1}{n}\left(\frac{\partial \log u}{\partial \bar{x}^j}\right)_o, \qquad f^o_{n+1} = \frac{1}{2} \bar{G}^{ij} f^o_i f^o_j \tag{5.2}$$

$$f^{n+1}_j = 0, \qquad f^{n+1}_{n+1} = \left(u^{\frac{2}{n}}\right)_o, \qquad f^j_{n+1} \bar{G}^{pq} f^j_p f^o_q,$$

where a subscript zero after a parenthesis indicates that the quantity within the parentheses is evaluated for $x = \bar{x}_o$. Then if $\bar{x}_o$ is an arbitrary point, there is determined at $\bar{x}_o$ a transformation of the form (4.6), i.e., a transformation of the enlarged conformal group. Moreover, the formulas show that if the transformation (5.1) is of the type (4.6), the transformation determined by it according to our rule is (5.1) itself. Thus we have at each point a multiple isomorphism between the totality of analytic transformations of coördinates and the enlarged conformal group of a Euclidean space.

6. *Conformal Tensors.*—This isomorphism established, it is clear that conformal tensors can be defined with a law of transformation in which the quantities f^α_β at each point play the same rôle as the derivatives $\delta x^i / \delta x^j$ in the law of transformation of affine tensors. But instead of the quantities f^α_β, we shall use the following quantities which differ from them by constant factors:

$$u^i_j = \frac{\partial x^i}{\partial \bar{x}^j} \qquad u = \left|\frac{\partial x}{\partial \bar{x}}\right|$$

$$u^\alpha_o = \delta^\alpha_o, \quad u^o_j = \frac{\partial \log u}{\partial \bar{x}^j}, \quad u^o_{n+1} = \frac{1}{2} \bar{G}^{ij} u^o_i u^o_j \tag{6.1}$$

$$u^{n+1}_j = 0, \; u^{n+1}_{n+1} = u^{\frac{2}{n}}, \; u^j_{n+1} = \bar{G}^{pq} u^j_p u^o_q.$$

The set of quantities so defined are the elements of the matrix given by T. Y. Thomas on page 356, Vol. 12 of these PROCEEDINGS.

Let us denote the quantities analogous to u^α_β which are determined by the inverse of (5.1), by v^α_β. Then we have

$$u^\alpha_\beta v^\beta_\gamma = \delta^\alpha_\gamma$$

and

$$u^i_j v^j_k = \delta^i_k .$$

We now define *a relative conformal tensor of weight N as an invariant whose* components in any two coördinate systems x and $\bar{x}$ are related by the law of transformation

$$\bar{T}^{\alpha\ldots}_{\gamma\ldots}{}^{\beta}_{\delta} = u^N T^{\sigma\ldots}_{\tau\ldots}{}^{\xi}_{\rho} u^\tau_\gamma \ldots u^\rho_\delta v^\alpha_\sigma \ldots v^\beta_\xi .$$

So, as a special case, a contravariant conformal vector of weight zero has the law of transformation

$$\bar{V}^\alpha = V^\sigma v^\alpha_\sigma .$$

The fundamental relative tensor G_{ij} whose law of transformation is (1.3) determines by the equations (3.3) a conformal tensor $G_{\alpha\beta}$ of weight $-2/n$: its law of transformation is

$$\bar{G}_{\alpha\beta} = u^{-\frac{2}{n}} G_{\sigma\tau} u^\sigma_\alpha u^\tau_\beta .$$

The conformal tensors satisfy the same laws of combination as the ordinary (affine) tensors. In addition, there are special rules due to the special form of the matrix (4.1). For example, the component of a covariant tensor for which all the subscripts are zero is a scalar of the same weight as the given tensor. Also, the component of any contravariant tensor of weight N and degree p for which the superscripts are all $n + 1$ is a scalar of weight $N + 2p/n$.

In order to have the affine, projective and conformal tensors fit together into a harmonious theory, I think it advisable to change the definition of the projective tensors used in my previous paper[1] so that u^0_j shall be given the same meaning as in (6.1) above. This merely changes some of the constants in the law of transformation. But it makes it possible to state a number of relationships between the projective and conformal tensors in a simple manner. For example, the components of any conformal covariant vector with indices 0, 1, 2, . . ., n are the components of a projective covariant vector. It also brings our notation into agreement with that used by T. Y. Thomas and by writers on five-dimensional relativity.

7. *The Conformal Gradient.*—There is a process analogous to covariant differentiation by which any conformal tensor determines another con-

formal tensor with one subscript more. But this involves an additional device which did not appear in the affine or the projective cases. We can illustrate it by the case in which the given tensor is a relative scalar of weight N.

The law of transformation of this scalar is

$$\bar{T} = T\,u^N .$$

On differentiating this we find equations from which we can infer that the quantities

$$T_{\omega'} = \frac{\partial T}{\partial x^j}\,\delta^j_{\omega} + N\,T\delta^0_{\omega}$$

in which the subscript ω runs over the values 0, 1, 2, . . ., n are $n + 1$ of the components of a conformal tensor. On account of the condition

$$u^{n+1}_{\omega} = 0,$$

the law of transformation of these components does not involve the $(n + 1)$st component. But the law of transformation of the latter does involve the other components. The $(n + 1)$st component could therefore be assigned arbitrarily in one coördinate system. We shall determine it by requiring the conformal vector which we are seeking to satisfy the invariant condition,

$$G^{\alpha\beta}\,T_{\alpha}T_{\beta} = 0.$$

On expanding, this becomes

$$-2\,T_0T_{n+1} + G^{ij}T_iT_j = 0$$

which determines T_{n+1}. Hence, we can define a conformal vector by the formula,

$$T_{,\alpha} = \frac{\partial T}{\partial x^j}\,\delta^i_{\alpha} + NT\,\delta^0_{\alpha} + \frac{1}{2NT}\,G^{ij}\frac{\partial T}{\partial x^i}\frac{\partial T}{\partial x^j}\,\delta^{n+1}_{\alpha}. \qquad (7.1)$$

This invariant will be called the *conformal gradient* of the relative scalar T. The formula fails when $N = 0$.

8. *The Extended Conformal Connection.*—In order to generalize covariant differentiation to the other conformal tensors we need another invariant with which to eliminate the higher derivatives which appear in the law of transformation of the derivatives of a conformal tensor. This invariant can be obtained by direct formal generalization from the affine and projective cases. The law of transformation,

$$\bar{K}^{\alpha}_{\beta\omega} = K^{\rho}_{\sigma\psi}\,v^{\alpha}_{\rho}\,u^{\sigma}_{\beta}\,u^{\psi}_{\omega} + \frac{\partial u^{\beta}_{\beta}}{\partial \bar{x}^j}\,\delta^j_{\omega}\,v^{\alpha}_{\rho} \qquad (8.1)$$

is transitive, provided the superscript and the first subscript run over the values 0, 1, 2, . . ., $n + 1$ whereas the second subscript (ω, ψ) runs over the indices 0, 1, 2, . . ., n only. Hence, the $(n + 2)^2(n + 1)$ functions $K^{\alpha}_{\beta\omega}$ are the components of an invariant which we shall call an extended conformal connection. This name is justified by the fact that the n^3 components K^{i}_{jk} are the components of a conformal connection in the sense of §2, provided that the extended conformal connection satisfies the invariant conditions,

$$K^{\alpha}_{o\beta} = K^{\alpha}_{\beta o} = -\frac{1}{n}\delta^{\alpha}_{\beta} \quad \text{and} \quad K^{n+1}_{ij} = -\frac{1}{n}G_{ij}. \qquad (8.2)$$

The law of transformation (8.1) is the same as that of the "associated connection" defined by T. Y. Thomas, Vol. **12** of these PROCEEDINGS, p. 357. It is also related to the conformal connection considered by Schouten.[6]

Given any conformal tensor of weight N let us differentiate its law of transformation and eliminate the second derivative between the resulting formula and the law of transformation (8.1) of the extended conformal connection. If the given tensor is a conformal covariant vector T_{α}, for example, the result of this elimination is the set of equations which state that

$$T_{\alpha,\omega} = \frac{\partial T_{\alpha}}{\partial x^{j}}\delta^{j}_{\omega} + NT_{\alpha}\delta^{o}_{\omega} - K^{\beta}_{\alpha\omega}T_{\beta}$$

are components of a covariant conformal tensor.

Here again the subscript ω runs only over the indices 0, 1, 2, . . ., n. In order to determine the remaining components of the tensor which we are seeking we employ the invariant condition

$$G^{\alpha\beta}T_{o,\alpha}T_{o,\beta} = 0$$

which gives the formula

$$2T_{o,o}T_{o,n+1} = G_{ij}T_{o,i}T_{o,j}$$

and determines the component $T_{o,n+1}$. We then employ the condition

$$G^{\alpha\beta}T_{o,\alpha}T_{\gamma,\beta} = 0$$

which determines the rest of the components, for it expands into

$$T_{o,o}T_{\gamma,n+1} = -T_{o,n+1}T_{\gamma,o} + G^{ij}T_{o,i}T_{\gamma,j}.$$

The conformal tensor $T_{\alpha,\gamma}$ which is thus defined we call the *conformal derivative* of T_{α}.

If the given tensor is a contravariant vector of weight N, we differentiate

its law of transformation and by eliminating with (8.1) find that the quantities

$$T^{\alpha}_{,\omega} = \frac{\partial T^{\alpha}}{\partial x^{j}} \delta^{j}_{\omega} + NT^{\alpha} \delta^{o}_{\omega} + K^{\alpha}_{\beta\omega} T^{\beta}$$

in which α runs from 0 to $n + 1$ and ω from 0 to n are components of a conformal tensor. The remaining components of this tensor are determined by the invariant condition,

$$0 = G^{\sigma\tau} T^{\alpha}_{,\sigma} T^{\beta}_{,\tau} = -T^{\alpha}_{,o} T^{\beta}_{,n+1} - T^{\alpha}_{,n+1} T^{\beta}_{,o} + G^{ij} T^{\alpha}_{,i} T^{\beta}_{,j}.$$

For $\alpha = \beta = n + 1$, this gives

$$T^{n+1}_{,n+1} = \frac{G^{ij} T^{n+1}_{,i} T^{n+1}_{,j}}{2T^{n+1}_{,o}}$$

and for α general and $\beta = n + 1$, it gives

$$T^{\alpha}_{,n+1} = \frac{G^{ij} T^{\alpha}_{,i} T^{n+1}_{,j} - T^{\alpha}_{,o} T^{n+1}_{,n+1}}{T^{n+1}_{,o}}.$$

The conformal tensor $T^{\alpha}_{,\beta}$, now fully defined, is called the *conformal derivative* of T^{α}.

It is now easy to write the formulas for the conformal derivative of any conformal tensor. For example, in case of a covariant tensor $T_{\alpha\beta}$ of weight N we have

$$T_{\alpha\beta,\omega} = \frac{\partial T_{\alpha\beta}}{\partial x^{j}} \delta^{j}_{\omega} + NT_{\alpha\beta} \delta^{o}_{\omega} - K^{\sigma}_{\alpha\omega} T_{\alpha\beta} - K^{\sigma}_{\beta\omega} T_{\alpha\sigma}$$

where ω runs from 0 to n, and

$$T_{oo,n+1} = \frac{G^{ij} T_{oo,j} T_{oo,j}}{2T_{oo,o}}$$

and

$$T_{\alpha\beta,n+1} = \frac{G^{ij} T_{oo,i} T_{\alpha\beta,j} - T_{\alpha\beta,o} T_{oo,n+1}}{T_{oo,o}}.$$

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[1] O. Veblen, these PROCEEDINGS, **14** (1928), p. 154.

[2] T. Y. Thomas, *Ibid.*, **11** (1925), p. 772.

[3] *Ibid.*, **12** (1926), p. 352.

[4] J. M. Thomas, these PROCEEDINGS, **11** (1925), p. 257.

[5] *Ibid.*, **12** (1926), p. 389.

[6] J. A. Schouten, *Verslagen Koninklijke Akademie van Wetenschappen te Amsterdam*, **27** (English) (1924), p. 407, and *Rendiconti del Circolo Matematico di Palermo*, **50** (1926).

[7] H. Weyl, *Göttinger Nachrichten*, **1921**, p. 99.

[8] E. Cartan, *Annales de la Société polonaise de mathématiques*, **1923**, p. 171.

THE GROUP MANIFOLD OF FINITE CONTINUOUS POINT AND FUNCTIONAL TRANSFORMATION GROUPS

By Aristotle D. Michal

Department of Mathematics, The Ohio State University

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1. *Introduction.*—The differential geometry of the manifold of the r parameters $a^1, a^2, \ldots, a^r$ of an r-parameter continuous group of transformations

$$\bar{x}^i = \varphi^i (x^1, x^2, \ldots, x^n; a^1, a^2, \ldots, a^r) \quad (i = 1, 2, \ldots, n)$$

has recently been studied by Cartan[1] with the aid of ingenious *geometric* methods peculiar to his way of looking at an affinely connected manifold.[2] From the analytic and invariant-theoretic standpoint, however, Cartan's geometric methods are by no means the last word on the subject. Furthermore, the proof of many of his theorems bring into play the variables $x^1, x^2, \ldots, x^n$. It is the purpose of this paper to develop some of the essential results of Cartan's theory analytically in such a way as to make the theory applicable to finite continuous groups of functional transformations in function space as well as to point transformations in n-dimensional space, and to exhibit various fundamental tensor invariants of the manifold in an *arbitrary coördinate system.*

A simple example of an r-parameter group of functional transformations is given by

$$y(x) = y(x) + \int_o^x K(x,s; a^1, a^2, \ldots, a^r) y(s) ds$$

where

$$K(x,s; a^1, a^2, \ldots, a^r) = \sum_{i=1}^{r} a^i H_i(x,s) + \frac{1}{2!} \sum_{i,j=1}^{r} a^i a^j \int_x^s H_i(x,u) H_j(u,s) du + \ldots + \ldots$$

and

$$\int_x^s H_i(x,u) H_j(u,s) du - \int_x^s H_j(x,u) H_i(u,s) du = \sum_{k=1}^{r} C_{ij}^k H_k(x,s) \quad (i,j = 1,2,\ldots,r).$$

What we have to say will apply equally well to finite continuous groups G in any abstract space for which a continuous group theory is possible. The central feature of all these group theories is found in their possession of an r-dimensional manifold: the manifold of the r-essential parameters

of the group. We shall call this manifold the *group manifold* of the group G.

2. *The Three Coefficients of Connection.*—The allowable point transformations in the group manifold are given by either of the two parameter-groups R_1 and R_2 of G

$$R_1: \bar{a}^i = f^i(a^1, a^2, \ldots, a^r; b^1, b^2, \ldots, b^r) \ (i = 1, 2, \ldots, r) \tag{1}$$

$$R_2: \bar{a}^i = f^i(b^1, b^2, \ldots, b^r; a^1, a^2, \ldots, a^r) \ (i = 1, 2, \ldots, r). \tag{2}$$

The two parameter-groups R_1 and R_2 essentially translate the fact that the ∞^r transformations in G actually form a group.

Let[3]

$$U_\alpha f = \xi^i_{(\alpha)} \frac{\partial f}{\partial a^i} \tag{3}$$

be the r infinitesimal generators[4] of the parameter-group R_1.

The r vectors $\xi^i_{(1)}, \xi^i_{(2)}, \ldots, \xi^i_{(r)}$ are linearly independent with respect to function multipliers since R_1 is a simple transitive group.[5] Thus the determinant

$$\Delta = |\xi^i_{(\alpha)}| \neq 0. \tag{4}$$

If we let $\Delta^{(\alpha)}_i$ denote the cofactor of $\xi^i_{(\alpha)}$ in Δ divided by Δ, we shall have obviously

$$\xi^i_{(\alpha)} \Delta^\beta_i = \delta^\beta_\alpha = \xi^\beta_{(i)} \Delta^{(i)}_\alpha, \tag{5}$$

where δ^β_α is zero or unity according as $\alpha \neq \beta$ or $\alpha = \beta$.

Since $\Delta^{(\beta)}_i$ is a covariant vector for each β, it follows that the set of functions $P^i_{\alpha\beta}$ defined by

$$P^i_{\alpha\beta} = \xi^i_{(\gamma)} \frac{\partial \Delta^{(\gamma)}_\alpha}{\partial a^\beta} \left(= - \Delta^{(\gamma)}_\alpha \frac{\partial \xi^i_{(\gamma)}}{\partial a^\beta} \right) \tag{6}$$

are the components of an asymmetric linear connection.[6] It is well known that the infinitesimal generators $U_\alpha f$ satisfy the identities[4]

$$(U_\alpha, U_\beta) f = C^\gamma_{\alpha\beta} U_\gamma f. \tag{7}$$

In other words the contravariant vectors $\xi^i_{(\alpha)}$ satisfy the relations

$$\xi^\sigma_{(\alpha)} \frac{\partial \xi^i_{(\beta)}}{\partial a^\sigma} - \xi^\sigma_{(\beta)} \frac{\partial \xi^i_{(\alpha)}}{\partial a^\sigma} = C^\gamma_{\alpha\beta} \xi^i_{(\gamma)}. \tag{8}$$

The quantities $C^\gamma_{\alpha\beta}$ in (8) are the structural constants of the group R_1 and thus satisfy the identities

$$\begin{cases} C^\gamma_{\alpha\beta} + C^\gamma_{\beta\alpha} = 0 \\ C^\sigma_{\alpha\beta} C^i_{\gamma\sigma} + C^\sigma_{\gamma\alpha} C^i_{\beta\sigma} + C^\sigma_{\beta\gamma} C^i_{\alpha\sigma} = 0. \end{cases} \tag{9}$$

Let

$$V_\alpha f = \eta^i_{(\alpha)} \frac{\partial f}{\partial a^i} \tag{10}$$

be the infinitesimal generators of the second parameter-group R_2. The r vectors $\eta^i_{(\alpha)}$ are connected with the vectors $\xi^i_{(\alpha)}$ of R_1 by the reciprocity relations[7]

$$(U_\alpha, V_\beta)f = 0. \tag{11}$$

An equivalent form of these conditions is given by

$$\xi^k_{(\alpha)} \frac{\partial \eta^i_{(\beta)}}{\partial a^k} - \eta^k_{(\beta)} \frac{\partial \xi^i_{(\alpha)}}{\partial a^k} = 0. \tag{12}$$

If we let Φ stand for the determinant of the $\eta^i_{(\alpha)}$, and $\Phi^{(\alpha)}_i$ for the cofactor of $\eta^i_{(\alpha)}$ in Φ divided by Φ, then the set of functions $Q^i_{\alpha\beta}$ defined by

$$Q^i_{\alpha\beta} = \eta^i_{(\gamma)} \frac{\partial \Phi^{(\gamma)}_\alpha}{\partial a^\beta} \left(= - \Phi^{(\gamma)}_\alpha \frac{\partial \eta^i_{(\gamma)}}{\partial a^\beta} \right) \tag{13}$$

are the components of an asymmetric connection. *Thus in the group manifold there exist two asymmetric coefficients of connection:*[8] *the first corresponds to the parameter-group R_1 and the second to the parameter-group R_2.*

If we multiply both sides of (12) by $\Phi^{(\beta)}_\gamma \Delta^{(\alpha)}_\delta$ and sum on α and β, we arrive at the relation

$$-Q^i_{\gamma\delta} + P^i_{\delta\gamma} = 0.$$

We are thus led to the following fundamental theorem.

Theorem 1. If $P^i_{\alpha\beta}$ is the coefficient of connection formed from the first parameter-group R_1, then $Q^i_{\alpha\beta} = P^i_{\beta\alpha}$ is the coefficient of connection formed from the second parameter-group R_2.

The geometry of paths[9] determined by

$$\Gamma^i_{\alpha\beta} = {}^1/_2(P^i_{\alpha\beta} + P^i_{\beta\alpha}) \tag{14}$$

is thus the same for both parameter-groups R_1 and R_2.

3. *Curvature Tensors of Asymmetric Connections and the Torsion Tensor.*—Let

$$\Omega^i_{\alpha\beta} = {}^1/_2(P^i_{\alpha\beta} - P^i_{\beta\alpha}). \tag{15}$$

Thus it is clear that the torsion tensor[10] based on the asymmetric connection $Q^i_{\alpha\beta}$ is equal to $-\Omega^i_{\alpha\beta}$.

The torsion tensor Ω^i_{lm} can be expressed in terms of the group structural constants $C^\gamma_{\alpha\beta}$ and the vectors $\xi^i_{(\alpha)}$. In fact, on multiplying both sides of (8) by ${}^1/_2\Delta^{(\alpha)}_l \Delta^{(\beta)}_m$ and summing on α and β, we obtain the result

$$\Omega^i_{lm} = {}^1/_2 C^\gamma_{\alpha\beta} \xi^i_{(\gamma)} \Delta^{(\alpha)}_l \Delta^{(\beta)}_m. \tag{16}$$

By a judicious choice of coördinate systems a number of theorems and formulas can be established quite simply. For the purposes of this paper it is convenient to have a coördinate system $\bar{a}^i$ such that

$$(\bar{\xi}^i_{(\alpha)}(\bar{a}))\bar{a}^{j=0} = \delta^i_\alpha \text{ and } (\bar{\Gamma}^i_{\alpha\beta}(\bar{a}))\bar{a}^{j=0} = 0. \tag{17}$$

A coördinate system for which the relations (17) are satisfied will be called *a canonical coördinate system.*[11]

Such a canonical coördinate system $\bar{a}^i$ can be defined by the coördinate transformation

$$a^i = q^i + (\xi^i_{(\alpha)})_q\bar{a}^\alpha - {}^1/_2(\Gamma^i_{\lambda\mu}\xi^\lambda_{(\alpha)}\xi^\mu_{(\beta)})_q\bar{a}^\alpha\bar{a}^\beta. \tag{18}$$

For the sake of clarity we note that the point with coördinates $a^i = q^i$ is the origin of the special coördinate system. Since the parameter-group R_1 is simply transitive it follows that the Jacobian of the transformation (18) does not vanish at the point $a^i = q^i$ and hence (18) is a reversible transformation in the neighborhood of that point.

It is a consequence of (17) that

$$(P^i_{\alpha\beta})_o = -(P^i_{\beta\alpha})_o \; ; \; (\Delta^{(i)}_k)_o = \delta^i_k. \tag{19}$$

The symbol $(\;)_o$ in (19) is an abbreviation for the operation which evaluates the quantity within the parenthesis at the origin of a canonical coördinate system. The omission of the bars above the quantities in the parenthesis need not cause any confusion.

Theorem 2. The curvature tensors P^i_{jkl} and Q^i_{jkl} of the parameter-groups R_1 and R_2, respectively, vanish.

To prove the first part of this theorem we proceed as follows. By means of (19) and (16) we see that

$$(P^i_{\alpha\beta})_o = {}^1/_2C^i_{\alpha\beta} \tag{20}$$

and hence

$$\left(\frac{\partial\xi^i_{(\alpha)}}{\partial a^\beta}\right)_o = -\frac{1}{2}C^{\;i}_{\alpha\beta} \; ; \; \left(\frac{\partial\Delta^{(i)}_\alpha}{\partial a^\beta}\right)_o = \frac{1}{2}C^i_{\alpha\beta}. \tag{21}$$

But from the form of the curvature tensor P^i_{jkl} we have

$$(P^i_{jkl})_o = \left(\frac{\partial P^i_{jk}}{\partial a^l} - \frac{\partial P^i_{jl}}{\partial a^k}\right)_o.$$

Evaluating this relation with the aid of (6) and (21) we get

$$(P^i_{jkl})_o = -{}^1/_4C^i_{al}\,C^a_{jk} + {}^1/_4\,C^i_{ak}C^a_{jl}. \tag{22}$$

Consequently,

$$(P^i_{jkl})_o = 0.$$

Since P^i_{jkl} is a tensor, we obtain by a familiar reasoning the result

$$P^i_{jkl} = 0.$$

By calculation we have

$$\left(\frac{\partial P^i_{kj}}{\partial a^l} - \frac{\partial P^i_{lj}}{\partial a^k}\right)_o = \frac{3}{4}C^a_{kj}\, C^i_{la} + \frac{3}{4}C^a_{jl}\, C^i_{ka} + C^a_{kl}\, C^i_{aj}. \qquad (23)$$

This relation is obtained by first substituting the expressions for P^i_{kj} and P^i_{lj} in terms of P^i_{jk} and P^i_{jl}, respectively, by means of (16) and then by employing the results in (22) and (21). Finally, by (20) and the form of the curvature tensor Q^i_{jkl} we obtain

$$(Q^i_{jkl})_o = C^a_{kj}\, C^i_{la} + C^a_{lk}\, C^i_{ja} + C^a_{jl}\, C^i_{ka}$$

and hence without difficulty

$$Q^i_{jkl} = 0.$$

Theorem 3. All three covariant derivatives of the torsion tensor Ω^i_{jk} based on the affine connections P^a_{bc}, Q^a_{bc}, Γ^a_{bc}, respectively, are zero.

On differentiating the equation (16) with respect to a^l in a system of canonical coördinates and evaluating the resultant expression at the origin of this coördinate system we obtain the relation

$$\left(\frac{\partial \Omega^i_{jk}}{\partial a^l}\right)_o = -\frac{1}{4}C^a_{jk}\, C^i_{al} + \frac{1}{4}C^i_{ak}\, C^a_{jl} + \frac{1}{4}C^i_{ja}\, C^a_{kl}.$$

Taking cognizance of the fact that $(\Gamma^i_{\alpha\beta})_o = 0$, it is easy to arrive at the result

$$\Omega^i_{jk;l} = 0,$$

where $\Omega^i_{jk;l}$ denotes the covariant derivative of Ω^i_{jk} based on the symmetric connection Γ^a_{bc}. On the basis of these results and the fact that $(P^i_{\alpha\beta})_o = {}^1/_2 C^i_{\alpha\beta}$, one can show that the covariant derivatives of Ω^i_{jk} based on the asymmetric connections $P^i_{\alpha\beta}$ and $Q^i_{\alpha\beta}$ also vanish.

4. The Curvature Tensor of the Symmetric Connection.—Let us now prove the following theorem.

Theorem 4. The covariant derivative B^i_{jklm} of the curvature tensor B^i_{jkl} based on the symmetric affine connection $\Gamma^i_{\alpha\beta}$ vanishes.

To prove this theorem we first observe that

$$(B^i_{jkl})_o = \left(\frac{\partial \Gamma^i_{jk}}{\partial a^l} - \frac{\partial \Gamma^i_{jl}}{\partial a^k}\right)_o.$$

With the aid of (22), (23) and the identities (9) satisfied by the structural constants $C^i_{\alpha\beta}$, we are led to the formula

$$(B^i_{jkl})_o = {}^1/_4 C^i_{aj}\, C^a_{kl}. \qquad (24)$$

From the transformation (18) we see that

$$\left.\begin{aligned} \frac{\partial a^i}{\partial \bar{a}^k} &= \xi^i_{(k)} \text{ when } a^i = q^i \\ \frac{\partial \bar{a}^i}{\partial a^k} &= \Delta^{(i)}_k \text{ when } a^i = q^i. \end{aligned}\right\} \tag{25}$$

The tensor law of transformation of B^i_{jkl} coupled with the results (24), (25) and the fact that the origin of canonical coördinates is arbitrary leads us to the following expression for the curvature tensor in an arbitrary coördinate system

$$B^i_{jkl} = {}^1/_4 C^\sigma_{\rho\alpha}\, C^\rho_{\beta\gamma}\, \xi^i_{(\sigma)}\, \Delta^{(\alpha)}_j\, \Delta^{(\beta)}_k\, \Delta^{(\gamma)}_l. \tag{26}$$

On differentiating equations (26) with respect to a^m when the equations are referred to a system of canonical coördinates a^i and evaluating the resultant expression at the origin of this system, we obtain by virtue of (21) and the identities (9) the fact that

$$\left(\frac{\partial B^i_{jkl}}{\partial a^m}\right)_o = 0.$$

The truth of theorem 4 is now obvious.

5. *Replacement Theorem.*—The purpose of this paragraph is to prove the theorem.

Theorem 5. Any tensor differential invariant (relative or absolute)

$$T^{\lambda\mu\ldots\nu}_{\rho\sigma\ldots\tau}\left(\xi^i_{(\alpha)}, \frac{\partial \xi^i_{(\alpha)}}{\partial a^j}, \frac{\partial^2 \xi^i_{(\alpha)}}{\partial a^j \partial a^k}\right)$$

of the group vectors $\xi^i_{(\alpha)}$ can by a mere replacement of the arguments be put into the form

$$T^{\lambda\mu\ldots\nu}_{\rho\sigma\ldots\tau}\left(\xi^i_{(\alpha)}, \xi^i_{(\alpha)j}, \xi^i_{(\alpha)jk}\right)$$

where

$$\xi^i_{(\alpha)j} = {}^1/_2 C^\lambda_{\sigma\alpha}\, \xi^i_{(\lambda)}\, \Delta^{(\sigma)}_j \tag{27}$$

and

$$\xi^i_{(\alpha)jk} = {}^1/_{12}(C^\gamma_{\alpha\lambda}\, C^\sigma_{\mu\gamma} + C^\gamma_{\alpha\mu}\, C^\sigma_{\lambda\gamma})\xi^i_{(\sigma)}\, \Delta^{(\lambda)}_j\, \Delta^{(\mu)}_k. \tag{28}$$

To prove this theorem we first need to show that the first tensor extension $\xi^i_{(\alpha)j}$ (covariant derivative based on $\Gamma^i_{\alpha\beta}$) of $\xi^i_{(\alpha)}$ is given by formula (27). In a canonical coördinate system

$$\left(\frac{\partial \xi^i_{(\alpha)}}{\partial a^j}\right)_o = \frac{1}{2} C^i_{j\alpha}.$$

Since $\xi^i_{(\alpha)j}$ is a tensor such that

$$(\xi^i_{(\alpha)j})_o = \left(\frac{\partial \xi^i_{(\alpha)}}{\partial a^j}\right)_o$$

we can arrive at formula (27) by making use of an argument that was employed in deriving expression (26) for the curvature tensor B^i_{jkl}.

We next need to show that the second tensor extension $\xi^i_{(\alpha)jk}$ (not the second covariant derivative) of $\xi^i_{(\alpha)}$ is given by (28). By Veblen's *normal coördinate*[12] methods it is not difficult to show that

$$\xi^i_{(\alpha)jk} = \frac{\partial^2 \xi^i_{(\alpha)}}{\partial a^j \partial a^k} + \frac{1}{3}\,\xi^\sigma_{(\alpha)}\left(\frac{\partial \Gamma^i_{\sigma j}}{\partial a^k} + \frac{\partial \Gamma^i_{k\sigma}}{\partial a^j} + \frac{\partial \Gamma^i_{jk}}{\partial a^\sigma}\right) + * \qquad (29)$$

where the symbol * stands for terms that vanish with $\Gamma^i_{\alpha\beta}$. Now

$$\Gamma^i_{\sigma j} = -\frac{1}{2}\left(\Delta^{(\gamma)}_\sigma \frac{\partial \xi^i_{(\gamma)}}{\partial a^j} + \Delta^{(\gamma)}_j \frac{\partial \xi^i_{(\gamma)}}{\partial a^\sigma}\right)$$

and hence by calculation we find

$$\left(\xi^{(\alpha)}_\sigma \frac{\partial \Gamma^i_{\sigma j}}{\partial a^k}\right)_o = -\frac{1}{2}\left(\frac{\partial^2 \xi^i_{(\alpha)}}{\partial a^j \partial a^k}\right)_o - \frac{1}{2}\left(\frac{\partial^2 \xi^i_{(j)}}{\partial a^\alpha \partial a^k}\right)_o + \frac{1}{8} C^\gamma_{\alpha k}\, C^i_{\gamma j} + \frac{1}{8} C^\gamma_{jk}\, C^i_{\gamma\alpha}. \qquad (30)$$

Consequently,

$$\left[\xi^\sigma_{(\alpha)}\left(\frac{\partial \Gamma^i_{\sigma j}}{\partial a^k} + \frac{\partial \Gamma^i_{k\sigma}}{\partial a^j} + \frac{\partial \Gamma^i_{jk}}{\partial a^\sigma}\right)\right]_o = -\left(\frac{\partial^2 \xi^i_{(\alpha)}}{\partial a^j \partial a^k} + \frac{\partial^2 \xi^i_{(j)}}{\partial a^k \partial a^\alpha} + \frac{\partial^2 \xi^i_{(k)}}{\partial a^\alpha \partial a^j}\right)_o. \qquad (31)$$

Now recalling that

$$(B^i_{\alpha jk})_o = \frac{1}{4} C^i_{\sigma\alpha}\, C^\sigma_{jk},$$

i.e.,

$$\left(\frac{\partial \Gamma^i_{\alpha j}}{\partial a^k} - \frac{\partial \Gamma^i_{\alpha k}}{\partial a^j}\right)_o = \frac{1}{4} C^i_{\sigma\alpha}\, C^\sigma_{jk},$$

we obtain the relation

$$\left(\frac{\partial^2 \xi^i_{(k)}}{\partial a^\alpha \partial a^j}\right)_o = \left(\frac{\partial^2 \xi^i_{(j)}}{\partial a^\alpha \partial a^k}\right)_o + \frac{1}{2} C^i_{\gamma\alpha}\, C^\gamma_{jk} + \frac{1}{4} C^\gamma_{\alpha j}\, C^i_{\gamma k} + \frac{1}{4} C^\gamma_{\alpha k}\, C^i_{\gamma j}. \qquad (32)$$

On expressing $(\partial^2 \xi^i_{(j)}/\partial a^k \partial a^\alpha)_o$ and $(\partial^2 \xi^i_{(k)}/\partial a^\alpha \partial a^j)_o$ in terms of $(\partial^2 \xi^i_{(\alpha)}/\partial a^j \partial a^k)_o$ by means of (32), relations (29) and (31) yield without difficulty the following result

$$(\xi^i_{(\alpha)jk})_o = \frac{1}{12}\,(C^\gamma_{\alpha j}\, C^i_{k\gamma} + C^\gamma_{\alpha k}\, C^i_{j\gamma}). \qquad (33)$$

Formula (28) is an immediate consequence of this result on the basis of the method used in establishing formula (26).

The final step in the proof of theorem 5 consists in showing the validity of the replacement process. Since the procedure is identical with that used in the various replacement theorems of the geometry of paths,[13] we spare the reader the calculations.

6. *Integral Invariants.*[14]—The following theorem is of importance in the calculation of invariants of continuous groups by integrations in the group manifold.

Theorem 6. The functionals

$$\int_{S_r} \left| \Delta_\alpha^{(i)} \right| da^1\, da^2 \ldots da^r \tag{34}$$

and

$$\int_{S_r} \left| \Phi_\alpha^{(i)} \right| da^1\, da^2 \ldots da^r \tag{35}$$

are integral invariants of the parameter-groups R_2 and R_1, respectively.

A necessary and sufficient condition that (34) be an integral invariant of the parameter-group R_2 is that the determinant $\left| \Delta_\alpha^{(i)} \right|$ satisfy the system of partial differential equations

$$\eta_{(\alpha)}^i \frac{\partial f}{\partial a^i} + f \frac{\partial \eta_{(\alpha)}^i}{\partial a^i} = 0. \tag{36}$$

Clearly

$$\frac{\partial \left| \Delta_\mu^{(\lambda)} \right|}{\partial a^i} = \left| \Delta_\mu^{(\lambda)} \right| \frac{\partial \Delta_c^{(b)}}{\partial a^i} \xi_{(b)}^c$$

$$= - \left| \Delta_\mu^{(\lambda)} \right| \Delta_c^{(b)} \frac{\partial \xi_{(b)}^c}{\partial a^i}.$$

This result coupled with the relation (12) enables one to verify that the determinant $\left| \Delta_\mu^{(\lambda)} \right|$ is a solution of (36). Similar methods lead to the result that the determinant $\left| \Phi_\alpha^{(i)} \right|$ is a solution of the system

$$\xi_\alpha^i \frac{\partial f}{\partial a^i} + f \frac{\partial \xi_{(\alpha)}^i}{\partial a^i} = 0.$$

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[1] E. Cartan, *J. Math. p. Appl.*, **6** (1927), pp. 1–119.

[2] E. Cartan, *Ann. Ecole Normal*, **40** (1923), pp. 325–412.

[3] The repetition of an index in a term will be used to denote summation with respect to this index over the integral values 1, 2, . . ., r. We shall also understand that free indices can take on any integral value 1, 2, . . ., r.

[4] L. Bianchi, *Lezioni Sulla Teoria Dei Gruppi Continui Finiti Di Trasformazioni*, 1918.

[5] L. Bianchi, loc. cit., pp. 297–304.

[6] L. P. Eisenhart, these PROCEEDINGS, pp. 246–250.

[7] L. Bianchi, loc. cit., pp. 270–272, p. 300.

[8] L. P. Eisenhart, "Non-Riemannian Geometry," *Amer. Math. Soc. Colloq. Publ.*, **8** (1927), Chapter I, Paragraph 2.

[9] O. Veblen and T. Y. Thomas, *Trans. Amer. Math. Soc.*, **25** (1923), pp. 551–580.

[10] P. Appell, *Traité de Mécanique Rationnelle*, Vol. V (1926), p. 169.

[11] Such a system is clearly a special kind of geodesic coördinate system. We refrain from using this terminology, however, on the ground that the coefficients of the linear terms in (18) are not the Kronecker deltas.

[12] O. Veblen and T. Y. Thomas, loc. cit., pp. 569–573. For a delightful introduction to the general subject of tensor analysis the reader is referred to O. Veblen, *Invariants of Quadratic Differential Forms*, 1927.

[13] T. Y. Thomas and A. D. Michal, *Annals Math.*, **28** (1927), p. 199.

[14] A. D. Michal, *Trans. Amer. Math. Soc.*, **29** (1927), pp. 612–646.